# Basic Concepts

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# Motivation

Most problems in nature have several (possibly conflicting) objectives to be satisfied. Many of these problems are frequently treated as single-objective optimization problems by transforming all but one objective into constraints.

# What is a multiobjective optimization problem?

The **Multiobjective Optimization Problem** (MOP) (also called multicriteria optimization, multiperformance or vector optimization problem) can be defined (in words) as the problem of finding (Osyczka, 1985):

> a vector of decision variables which satisfies constraints and optimizes a vector function whose elements represent the objective functions. These functions form a mathematical description of performance criteria which are usually in conflict with each other. Hence, the term "optimize" means finding such a solution which would give the values of all the objective functions acceptable to the decision maker.

# A Formal Definition

The general Multiobjective Optimization Problem (MOP) can be formally defined as:

Find the vector $\vec{x}^* = [x_1^*, x_2^*, \ldots, x_n^*]^T$ which will satisfy the $m$ inequality constraints:

$$g_i(\vec{x}) \geq 0 \quad i = 1, 2, \ldots, m \tag{1}$$

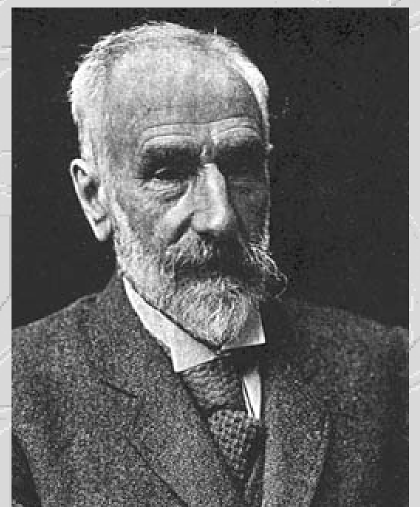
the $p$ equality constraints

$$h_i(\vec{x}) = 0 \quad i = 1, 2, \ldots, p \tag{2}$$

and will optimize the vector function

$$\vec{f}(\vec{x}) = [f_1(\vec{x}), f_2(\vec{x}), \ldots, f_k(\vec{x})]^T \tag{3}$$

# What is the notion of optimum in multiobjective optimization?



Having several objective functions, the notion of "optimum" changes, because in MOPs, we are really trying to find good compromises (or "trade-offs") rather than a single solution as in global optimization. The notion of "optimum" that is most commonly adopted is that originally proposed by Francis Ysidro Edgeworth in 1881.

# What is the notion of optimum in multiobjective optimization?

This notion was later generalized by Vilfredo Pareto (in 1896). Although some authors call *Edgeworth-Pareto optimum* to this notion, we will use the most commonly accepted term: *Pareto optimum*.

# Definition of Pareto Optimality

We say that a vector of decision variables $\vec{x}^* \in \mathcal{F}$ is *Pareto optimal* if there does not exist another $\vec{x} \in \mathcal{F}$ such that $f_i(\vec{x}) \leq f_i(\vec{x}^*)$ for all $i = 1, \ldots, k$ and $f_j(\vec{x}) < f_j(\vec{x}^*)$ for at least one $j$.

Here, $\mathcal{F}$ denotes the feasible region of the problem (i.e., where the constraints are satisfied).

## Definition of Pareto Optimality

In words, this definition says that $\vec{x}^*$ is Pareto optimal if there exists no feasible vector of decision variables $\vec{x} \in \mathcal{F}$ which would decrease some criterion without causing a simultaneous increase in at least one other criterion. Unfortunately, this concept almost always gives not a single solution, but rather a set of solutions called the *Pareto optimal set*. The vectors $\vec{x}^*$ correspoding to the solutions included in the Pareto optimal set are called *nondominated*. The plot of the objective functions whose nondominated vectors are in the Pareto optimal set is called the *Pareto front*.

# Pareto Dominance

A vector $\vec{u} = (u_1, \ldots, u_k)$ is said to dominate $\vec{v} = (v_1, \ldots, v_k)$ (denoted by $\vec{u} \preceq \vec{v}$) if and only if u is partially less than v, i.e., $\forall i \in \{1, \ldots, k\},\ u_i \leq v_i \wedge \exists i \in \{1, \ldots, k\} : u_i < v_i$.

## Pareto Optimal Set

For a given MOP $\vec{f}(x)$, the Pareto optimal set ($\mathcal{P}^*$) is defined as:

$$\mathcal{P}^* := \{x \in \mathcal{F} \mid \neg\exists\, x' \in \mathcal{F} \;\; \vec{f}(x') \preceq \vec{f}(x)\}. \tag{4}$$

## Pareto Front

For a given MOP $\vec{f}(x)$ and Pareto optimal set $\mathcal{P}^*$, the Pareto front ($\mathcal{PF}^*$) is defined as:

$$\mathcal{PF}^* := \{\vec{u} = \vec{f} = (f_1(x), \ldots, f_k(x)) \mid x \in \mathcal{P}^*\}. \tag{5}$$

# Convexity

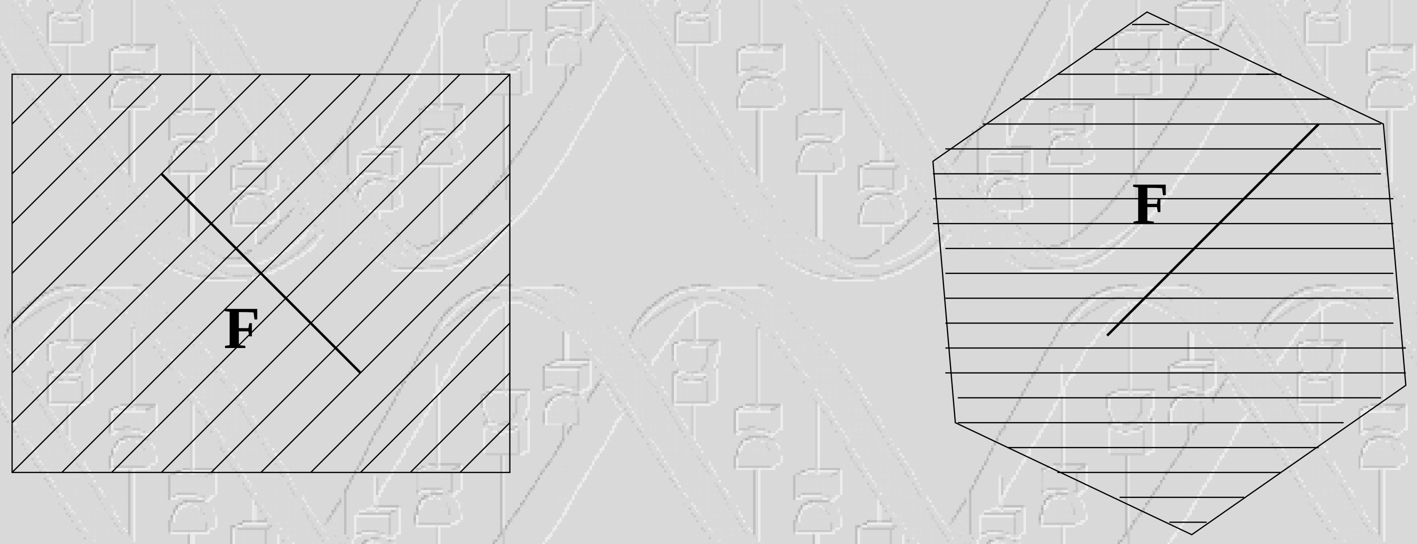



Figura 1: Examples of Convex Sets.

# Non Convexity

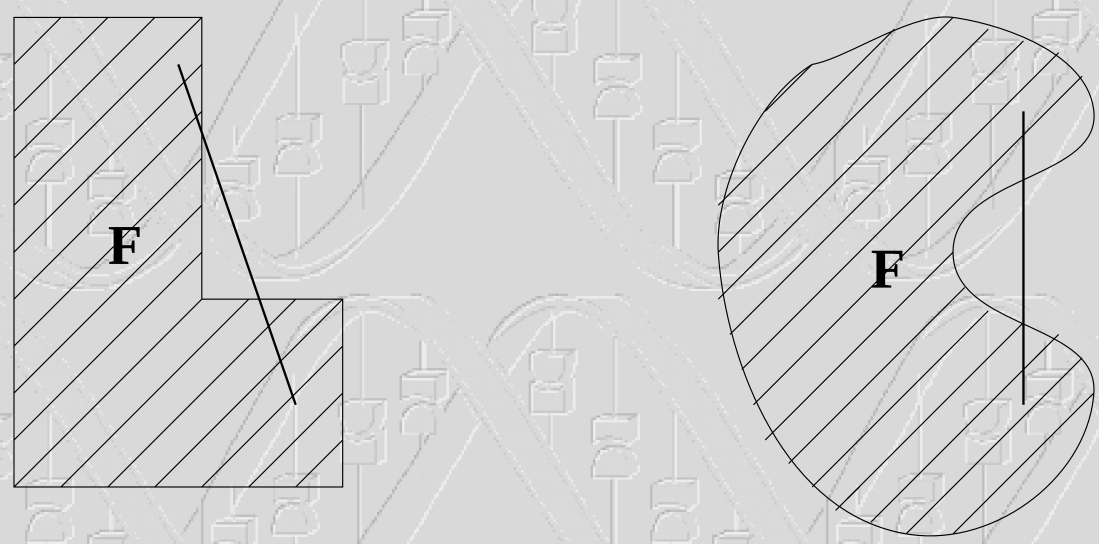



Figura 2: Examples of Non-Convex Sets.

## Ideal Vector

Vector containing the decision variables corresponding to the optima of the objective functions of the problems considering each objective separately.

# Weak Dominance

A point $\vec{x}^* \in \mathcal{F}$ is a **weakly nondominated solution** if there is no $\vec{x} \in \mathcal{F}$ such that $f_i(\vec{x}) < f_i(\vec{x}^*)$, for $i = 1, \ldots, k$.

## Strong Dominance

A point $\vec{x}^* \in \mathcal{F}$ is a **strongly nondominated solution** if there is no $\vec{x} \in \mathcal{F}$ such that $f_i(\vec{x}) \leq f_i(\vec{x}^*)$, for $i = 1, \ldots, k$ and for at least one value of $i$, $f_i(\vec{x}) < f_i(\vec{x}^*)$.

# Links to Operations Research

In Operations Research, it is a common practice to differentiate among attributes, criteria, objectives and goals:

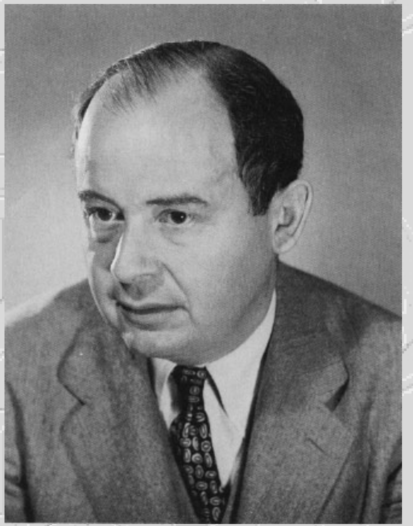
- **Attributes**: differentiating aspects, properties or characteristics of alternatives or consequences.
- **Criteria**: generally denote evaluative measures, dimensions or scales against which alternatives may be gauged in a value or worth sense.

## Links to Operations Research

- **Objectives**: are sometimes viewed in the same way as criteria, but may also denote specific desired levels of attainment or vague ideals.
- **Goals**: usually indicate either of the latter notions.

A distinction commonly made in Operations Research is to use the term *goal* to designate potentially attainable levels, and *objective* to designate unattainable ideals.

# Some Historical Highlights



As early as 1944, John von Neumann and Oskar Morgenstern mentioned that an optimization problem in the context of a social exchange economy was “a peculiar and disconcerting mixture of several conflicting problems” that was “nowhere dealt with in classical mathematics”.

# Some Historical Highlights

In 1951 Tjalling C. Koopmans edited a book called *Activity Analysis of Production and Allocation*, where the concept of "efficient" vector was first used in a significant way.

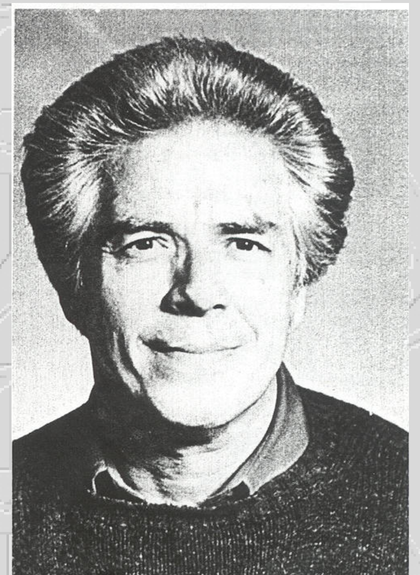
# Some Historical Highlights

The origins of the mathematical foundations of multiobjective optimization can be traced back to the period that goes from 1895 to 1906. During that period, Georg Cantor and Felix Hausdorff laid the foundations of infinite dimensional ordered spaces.

## Some Historical Highlights

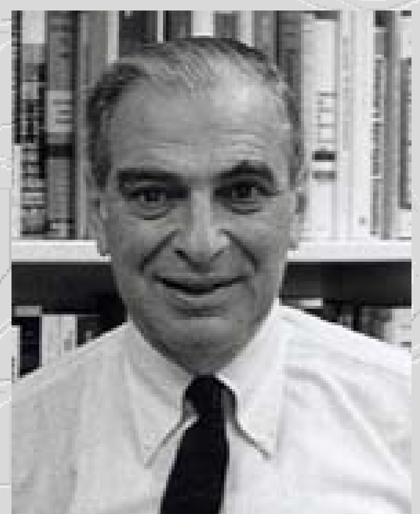
Cantor also introduced equivalence classes and stated the first sufficient conditions for the existence of a utility function. Hausdorff also gave the first example of a complete ordering.

# Some Historical Highlights



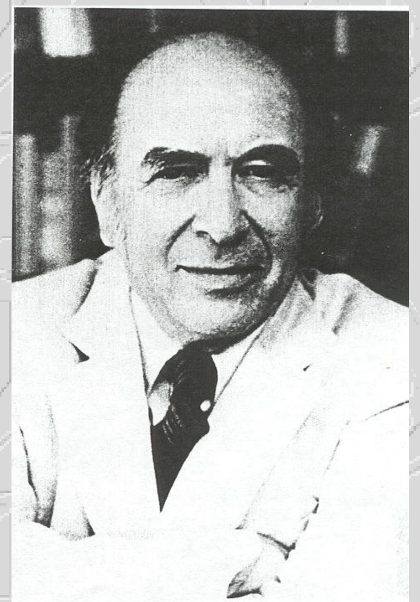
However, it was the concept of *vector maximum problem* introduced by Harold W. Kuhn and Albert W. Tucker (1951) which made multiobjective optimization a mathematical discipline on its own.

## Some Historical Highlights



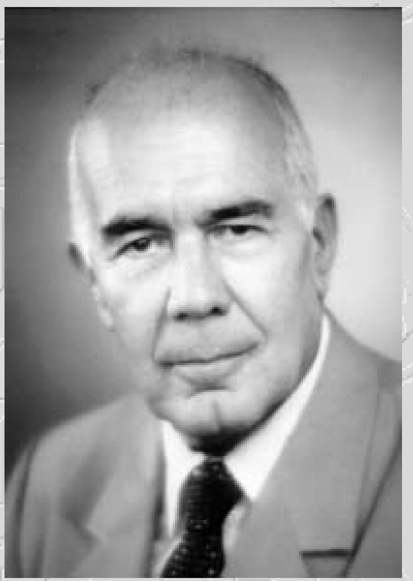
Kenneth J. Arrow did very important pioneering work during the 1950s, using the concept of admissible points and stating his famous theorem on multicriteria decision making (now called Arrow's impossibility theorem).

## Some Historical Highlights



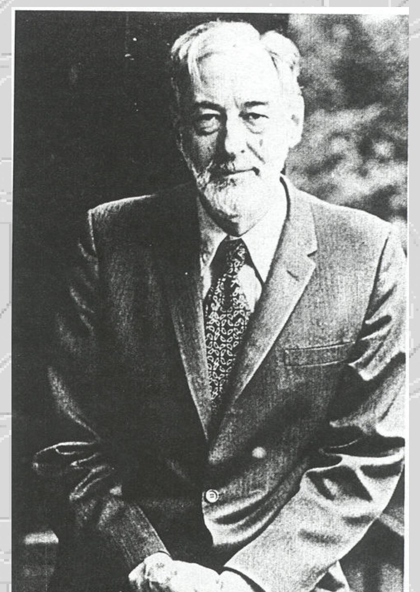
However, multiobjective optimization theory remained relatively undeveloped during the 1950s. It was until the 1960s that the foundations of multiobjective optimization were consolidated and taken seriously by pure mathematicians when Leonid Hurwicz generalized the results of Kuhn & Tucker to topological vector spaces.

## Some Historical Highlights



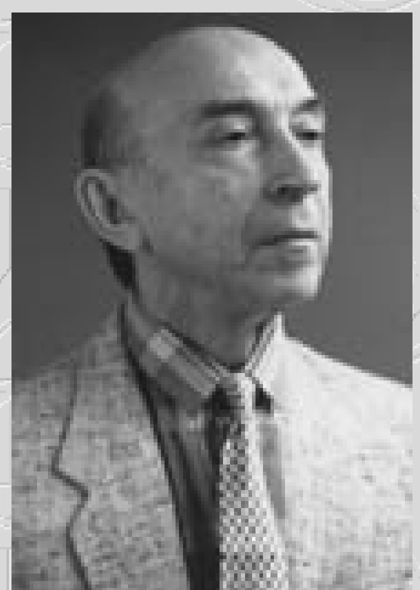
Perhaps the most important outcome of the 1950s was the development of the so-called *Goal Programming*, which was introduced by Abraham Charnes and William Wager Cooper in 1957.

# Some Historical Highlights



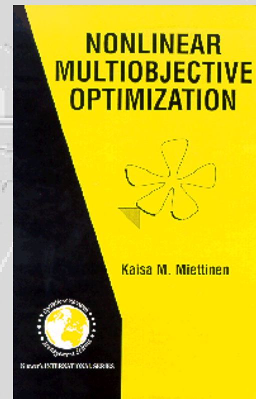
The application of multiobjective optimization to domains outside economics began with the work by Koopmans (1951) in production theory and with the work of Marglin (1967) in water resources planning.

# Some Historical Highlights



The first engineering application reported in the literature was a paper by Zadeh in the early 1960s. However, the use of multiobjective optimization became generalized until the 1970s.

# Current State of the Area




Currently, there are over 30 mathematical programming techniques for multiobjective optimization. However, these techniques tend to generate elements of the Pareto optimal set one at a time. Additionally, most of them are very sensitive to the shape of the Pareto front (e.g., they do not work when the Pareto front is concave or when the front is disconnected).

# Why Evolutionary Algorithms?

Evolutionary algorithms seem particularly suitable to solve multiobjective optimization problems, because they deal simultaneously with a set of possible solutions (the so-called population). This allows us to find several members of the Pareto optimal set in a single run of the algorithm, instead of having to perform a series of separate runs as in the case of the traditional mathematical programming techniques. Additionally, evolutionary algorithms are less susceptible to the shape or continuity of the Pareto front (e.g., they can easily deal with discontinuous or concave Pareto fronts), whereas these two issues are a real concern for mathematical programming techniques.