INCREASING SELECTIVE PRESSURE TOWARDS THE BEST COMPROMISE IN EVOLUTIONARY MULTIOBJECTIVE OPTIMIZATION: THE EXTENDED NOSGA METHOD.

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Abstract: Most current approaches in the evolutionary multiobjective optimization literature concentrate on adapting an evolutionary algorithm to generate an approximation of the Pareto frontier. However, finding this set does not solve the problem. The decision-maker still has to choose the best compromise solution out of that set. Here, we introduce a new characterization of the best compromise solution of a multiobjective optimization problem. By using a relational system of preferences based on a multicriteria decision aid way of thinking, and an outranked-based dominance generalization, we derive some necessary and sufficient conditions which describe satisfactory approximations to the best compromise. Such conditions define a lexicographic minimum of a bi-objective optimization problem, which is a map of the original one. The *NOSGA-II* method is a *NSGA-II* inspired efficient way of solving the resulting mapped problem.
Keywords: Evolutionary Algorithms; multiobjective optimization; multicriteria decision; preference modeling.

1. Introduction

In real-world optimization problems, the decision-maker (*DM*) is usually concerned with several criteria which determine the quality of solutions. Therefore, many optimization problems need to be represented from a multiple objective perspective.
As a consequence of the conflicting nature of the criteria, it is not possible to obtain a single optimum, and, consequently, the ideal solution of a multiobjective problem (*MOP*) cannot be reached. Unlike single-objective optimization, the best solution of a *MOP* is not well-defined (i.e., it is not defined from a purely mathematical point of view). To solve a *MOP* means to find the best compromise solution according to the *DM*´s particular system of preferences (value system). Since all the compromise solutions are mathematically equivalent, the *DM* should provide some additional information for choosing the most preferred one (cf. [15]). Such information can be provided before or after the optimization method generates compromise solutions, or the process can be interactive, performing a progressive articulation of the *DM*´s preferences [15].
Multi-Objective Evolutionary Algorithms (*MOEA*s) are particularly attractive to solve *MOP*s because they deal simultaneously with a set of possible solutions (the *MOEA*'s population) which allows them to obtain an approximation of the Pareto frontier in a single algorithm's run. Thus, by using *MOEA*s the *DM* and/or the decision analyst does not need to perform a set of separate single-objective optimizations (as normally required when using operations research methods) in order to generate compromise solutions. Additionally, *MOEA*s are more robust regarding the shape or continuity of the Pareto front, whereas these two issues are a real concern for operations research optimization methods (cf. [3]). Several types of MOEAs currently exist, ranging from those that adopt different variations of Pareto-based selection (see for example [29,30]) to the use of scalar subproblems which are simultaneously optimized (see for example [31]), with several

intermediate proposals that introduce clever modifications to well-known MOEAs that are aimed to improve their performance (see for example [32,33,34,35,36]). However, according to [5, 7, 10], one aspect that is often disregarded in the *MOEA*s' literature is the fact that the solution of a problem involves not only the search, but also the decision making process. Most current approaches in the evolutionary multiobjective optimization literature concentrate on adapting an evolutionary algorithm to generate an approximation of the Pareto optimal set (cf.[4]). Nevertheless, finding this set does not completely solve the problem. The *DM* still has to choose the best compromise solution out of that set ([5, 29] . This is not a difficult task when dealing with problems having 2 or 3 objectives. However, as the number of criteria increases, three important difficulties arise:

a) The algorithm's capacity to find this Pareto frontier quickly degrades (e.g. [28]);
b) It becomes harder, or even impossible for the *DM* to establish valid judgments in order to compare solutions with several conflicting criteria;
c) The cardinal of a representative portion of the known Pareto frontier may be too large; the approaches from the field of multicriteria decision analysis do not perform well on such large decision problems, making difficult to obtain a unique solution.

To overcome the above criticisms, in [10] we proposed the use of *a priori* articulation of preferences by creating a fuzzy outranking relation, followed by a generating process of a subset of the Pareto frontier. Using such a fuzzy outranking relation, a strict (crisp) preference relation is established on any population. In the *NOSGA* method (cf. [10]), that preference relation is used instead of dominance when performing the evolutionary search. In some 0-1 knapsack examples with 4-9 objectives, we obtained a privileged zone of the Pareto frontier, composed of relatively few, concentrated, and satisfactory solutions (cf. 10]). However, a typical *DM* is only able to process from five to nine pieces of knowledge at a time (cf. [18]), being thus unable to identify the best compromise solution when he/she needs to compare what still seems a relatively small subset of compromise solutions in problems having more than 5-9 objectives. In such cases, the progressive (interactive) and the *a posteriori* articulation of *DM*'s preferences can be very hard to use due to these human cognitive limitations. Improving the prior articulation of preferences becomes necessary in order to approach the best compromise among the objectives. The *MOEA*'s selective pressure towards the best compromise solution should be increased. But this is not possible without a good mathematical model of the concept of a *MOP*'s best compromise. Some recent proposals have been performed in order to incorporate *DM*'s preferences in multiobjective optimization (cf. [10, 26]). In this work, our previous proposal ([10]) is enhanced. The model of *DM*'s preferences is better than before, and a good theoretical characterization of the best compromise solution is achieved. Using this characterization, we improve convergence to a privileged zone on the Pareto frontier. This makes easier the solution of problems with many objectives.

The remainder of this paper is structured as follows: An appropriate concept of "best compromise solution" for a vector optimization problem is discussed in Section 2. A model of a relational system of preferences based on fuzzy outranking relations is presented in Section 3. Section 4 contains a bi-objective characterization of best compromise solutions. Supported by this background, our algorithmic proposal (*NOSGA-II*) is presented in Section

5, and it is illustrated by an example in Section 6. Finally, we present some concluding remarks.

2. The best compromise solution: What does it mean, exactly?

Let us consider a *MOP* of the form

$$\textbf{Maximize } \boldsymbol{F} = (\mathbf{f_1}(z), \mathbf{f_2}(z), \ldots \mathbf{f_n}(z)) \qquad (1)$$
$$z \in R_F$$

in which z denotes a vector of decision variables and R_F is determined by a set of constraints.

The action of maximizing in (1) is ill-defined. From a normative point of view, assuming the existence of a value function U ($f_1(z)$, $f_2(z)$,… $f_n(z)$) which agrees with the *DM*'s system of preferences, the "best" solution of (1) should be obtained by maximizing U on R_F (cf. [20, 26]). Unfortunately, the practical value of this statement is strongly limited for several reasons. The existence of such value functions is not guaranteed for real *DM*s (see [23, 24]) for a discussion of the practical limitations of decision actors). Moreover, even if the *DM* approached an ideal normative behavior, it would be extremely difficult, if not impossible, to specify his/her value function.

Other authors elude a formal definition. According to Ozyczka ([22]), solving (1) is to find a feasible solution which gives the values of all the objective functions acceptable to the *DM*. Thus, the best compromise solution is seen under an acceptability criterion, although an idea based on some kind of optimality should be more appropriate. In [16] Hwang and Masud define the best solution as a good compromise which is accepted by the *DM* as the final solution. According to this definition, the concept of best compromise is relative to the set of solutions which is generated by the algorithm and depends on the effort dedicated by the *DM* to compare compromise solutions. Coello et al. ([4]) state the need of selecting a compromise solution satisfying the objectives as "best" possible. Hakanan et al. ([15]) identify the best compromise as the compromise solution which is the most preferred one. Such statements are acceptable from an intuitive point of view, but a mathematical formalization is needed in order to make them useful by an evolutionary search.

In the following, we attempt to give a formal characterization of a best compromise solution to (1).

Let us denote by O the image of R_F in the objective space mapped by the vector function $\boldsymbol{F}$. An element $\boldsymbol{x} \in O$ is a vector (x_1, … x_n), where x_i is the i-th objective function value.

First of all, we define the concept of compromise solution: Let $\boldsymbol{w}$ be a Pareto solution for Problem (1); we say that $\boldsymbol{w}$ is a compromise solution if w_i reaches a minimum acceptable value for i=1,…n. Below, we introduce an operational approach of best compromise.

Let us suppose that the *DM* is comparing a representative set of compromise solutions suggested by some method for solving Problem (1). Let's also assume that $\boldsymbol{x^*}$ is a good compromise solution being considered by the *DM*. If the *DM* cannot identify other compromise solution which he/she judges to be (at least) slightly more satisfactory than $\boldsymbol{x^*}$, then this may be chosen as the final solution for Problem (1).

Definition 1: A compromise solution $\boldsymbol{x^*}$ is a best compromise solution in $C \subseteq O$ iff there is no $\boldsymbol{y} \in C$ such that the *DM* considers that " $\boldsymbol{y}$ is at least as good as $\boldsymbol{x^*}$" and simultaneously he/she rejects (or doubts about) the statement " $\boldsymbol{x^*}$ is at least as good as $\boldsymbol{y}$".

The above characterization matches with the normative definition based on a value function. If U is a value function for the *DM* and $U(\boldsymbol{x^*}) = max\ (U\ (\boldsymbol{x}))$, there is no $\boldsymbol{y}$ such that $U(\boldsymbol{y}) > U(\boldsymbol{x^*})$.

Let the predicate $S(\boldsymbol{x},\boldsymbol{y})$ be "the *DM* considers that option $\boldsymbol{x}$ is at least as good as $\boldsymbol{y}$" defined on $O \times O$. The logic negation of $S(\boldsymbol{x},\boldsymbol{y})$ (denoted by $.\ not\ S(\boldsymbol{x},\boldsymbol{y})$) corresponds to the statement "the *DM* disagrees (or partially disagrees) with $\boldsymbol{x}$ is at least as good as $\boldsymbol{y}$" . A conjunction $S(\boldsymbol{x},\boldsymbol{y}) \wedge .\ not\ S(\boldsymbol{y},\boldsymbol{x})$ is related to certain asymmetric preference favoring option $\boldsymbol{x}$ over $\boldsymbol{y}$. Besides, $.\ not\ S(\boldsymbol{x},\boldsymbol{y}) \wedge .\ not\ S(\boldsymbol{y},\boldsymbol{x})$ corresponds to a descriptive situation in which a real *DM* or decision actor cannot (or does not want to) make a decision when he/she is comparing $(\boldsymbol{x},\boldsymbol{y})$.These hesitations may come from any of the following reasons:

- the *DM* is a vaguely defined entity, or even a well-defined entity with poorly defined preference rules (cf. [24]);
- the existence in the *DM*'s mind (if the *DM* is a real person) of certain "zones" of uncertainty, imprecise beliefs, conflicts and competing aspirations ([24]);
- the existence of imprecise attribute values.

In [23] Roy described situations concerning this non-ideal behavior from real decision actors by using a relational system of preferences composed of several binary relations. The definitions from [23] are given below:

1. **Indifference**: It corresponds to the existence of clear and positive reasons that justifies equivalence between the two actions. Notation: $\boldsymbol{xIy}$.
2. **Strict preference**: It corresponds to the existence of clear and positive reasons that justify significant preference in favor of one (identified) of the two actions. The statement $\boldsymbol{x}$ is strictly preferred to $\boldsymbol{y}$ is denoted by $\boldsymbol{xPy}$. P is asymmetric and non reflexive.
3. **Weak preference**: It corresponds to the existence of clear and positive reasons in favor of $\boldsymbol{x}$ over $\boldsymbol{y}$, but that are not sufficient to justify strict preference. Indifference and strict preference cannot be distinguished appropriately. This is denoted by $\boldsymbol{x}\ Q\boldsymbol{y}$. Q is asymmetric and non reflexive.
4. **Incomparability**: None of the preceding situations predominates. That is, absence of clear and positive reasons that justify any of the above relations. Notation: $\boldsymbol{xRy}$. R is symmetric.
5. **Outranking**: It corresponds to the existence of clear and positive reasons that justify the statement "$\boldsymbol{x}$ is at least as good as $\boldsymbol{y}$", but with no significant division being established among the situations of strict preference, weak preference and indifference. Notation: $\boldsymbol{xSy}$.
6. ***K*-preference**: It corresponds to the existence of clear and positive reasons that justify strict preference in favor of one (identified) of the two objects or incomparability between the two objects, but with no significant division

established between the situations of strict preference and incomparability. Notation: $\boldsymbol{xKy}$. K is asymmetric.

7. **Nonpreference**: It corresponds to situations in which indifference and incomparability are both possible, without being able to differentiate between them. This is denoted by $\boldsymbol{x} \sim \boldsymbol{y}$. $\sim$ is symmetric.

Let us introduce an asymmetric preference relation $A_P = P \cup Q \cup K$. Combining A_P with Definition 1 we can suggest another characterization of the final solution for Problem (1).

Definition 2: (second characterization of a best compromise): Let C be a subset of O. Suppose that for each $(\boldsymbol{x},\boldsymbol{y}) \in C \times C$ one and only one of the following statements is true:

i) $\boldsymbol{x}A_P\boldsymbol{y}$
ii) $\boldsymbol{y}A_P\boldsymbol{x}$
iii) $\boldsymbol{x}I\boldsymbol{y}$
iv) $\boldsymbol{x}R\boldsymbol{y}$
v) $x \sim \boldsymbol{y}$

$\boldsymbol{x^*}$ is a best compromise solution in C iff there is no $\boldsymbol{y} \in C$ such that $\boldsymbol{y}A_P\, \boldsymbol{x^*}$.

Note that several best compromises may exist on C. Besides, a best compromise may not exist on C. That possibility arises when each pair $(\boldsymbol{x},\boldsymbol{y}) \in C \times C$ belongs to some cycle of A_P. When the set C is the whole objective space O, the above characterization may be used to approach a suitable most satisfactory solution for Problem (1). This should be one of the best compromise solutions in O. In practical situations, the *DM* always expresses his/her preferences on a proper subset of O. So, in fact it is not possible to guarantee that the final solution obtained on C is the real best solution. A best compromise obtained on $C \subset O$ is an approximation to the best solution of (1). Nevertheless, when C is representative of the satisfactory zone of the Pareto frontier, a best compromise defined on C may be sufficiently close to the best solution of (1).

3. A model of the relational system of preferences based on fuzzy outranking relations.

The model of *DM*'s preferences may be enhanced by considering $S(\boldsymbol{x},\boldsymbol{y})$ as a fuzzy predicate. Fuzzy binary preference relations are a good compromise between value functions and crisp preference relations; fuzzy relations are numerical, such as value functions, but their power of expressivity is higher since they can easily model incomparability and non-transitivity (cf. [13]).

In the following, we consider that there is a method for assigning a degree of truth $\sigma(\boldsymbol{x},\boldsymbol{y})$ in [0, 1] to the predicate $S(\boldsymbol{x},\boldsymbol{y})$. Outranking methods such as ELECTRE-III (cf. [24, 25]) and PROMETHEE (cf. [2]) may be used. Once $\sigma(\boldsymbol{x},\boldsymbol{y})$ has been calculated, it can be useful for modelling the crisp preference relations defined in Section 2. Let us consider $\lambda > 0.5$ a threshold of acceptable credibility for the S predicate. Let's consider also an asymmetry parameter β and a symmetry parameter ε $(0<\varepsilon<\beta<\lambda)$. A strict preference relation $\boldsymbol{x}P(\lambda,\beta)\boldsymbol{y}$ can be justified if at least one of the following conditions is held:

i. $\boldsymbol{x}$ dominates $\boldsymbol{y}$

ii. $\sigma(\boldsymbol{x},\boldsymbol{y}) \geq \lambda \wedge \sigma(\boldsymbol{y},\boldsymbol{x}) < 0.5$
iii. $\sigma(\boldsymbol{x},\boldsymbol{y}) \geq \lambda \wedge (0.5 \leq \sigma(\boldsymbol{y},\boldsymbol{x}) < \lambda) \wedge (\sigma(\boldsymbol{x},\boldsymbol{y}) - \sigma(\boldsymbol{y},\boldsymbol{x})) \geq \beta$

An indifference relation $\boldsymbol{x}I(\lambda,\varepsilon)\boldsymbol{y}$ can be justified if I1 and I2 are both satisfied:
I1. $\sigma(\boldsymbol{x},\boldsymbol{y}) \geq \lambda \wedge \sigma(\boldsymbol{y},\boldsymbol{x}) \geq \lambda$
I2. $|\sigma(\boldsymbol{x},\boldsymbol{y}) - \sigma(\boldsymbol{y},\boldsymbol{x})| \leq \varepsilon$

A weak preference relation $\boldsymbol{x}Q(\lambda,\beta,\varepsilon)\boldsymbol{y}$ is a consequence of the conjunction of three propositions:

A. $\sigma(\boldsymbol{x},\boldsymbol{y}) \geq \lambda \wedge \sigma(\boldsymbol{x},\boldsymbol{y}) > \sigma(\boldsymbol{y},\boldsymbol{x})$
B. $\boldsymbol{x}$. *not* $P(\lambda,\beta)\boldsymbol{y}$
C. $\boldsymbol{x}$ **.** *not* $I(\lambda,\varepsilon)\boldsymbol{y}$

A *K*- preference relation $\boldsymbol{x}K(\lambda,\beta)\boldsymbol{y}$ is modeled by the conjunction of the following three propositions:

A1. $0.5 \leq \sigma(\boldsymbol{x},\boldsymbol{y}) < \lambda$
B1. $\sigma(\boldsymbol{y},\boldsymbol{x}) < 0.5$
C1. $(\sigma(\boldsymbol{x},\boldsymbol{y}) - \sigma(\boldsymbol{y},\boldsymbol{x})) > \beta/2$
where Condition C1 has been included in order to reflect certain asymmetry which justifies a sort of preference favoring $\boldsymbol{x}$. This agrees with Point 6 of Section 2.

The incomparability relation $\boldsymbol{x}R\boldsymbol{y}$ is defined by $\sigma(\boldsymbol{x},\boldsymbol{y}) <0.5 \wedge \sigma(\boldsymbol{y},\boldsymbol{x}) <0.5$.

Definition 3: Let $\mathscr{P}$ denote a specific settlement of parameters λ, β, ε ($\lambda>0.5>\beta>\varepsilon>0$). We say that $\mathscr{P}$ is preferentially consistent iff $P(\lambda,\beta)$, $I(\lambda,\varepsilon)$, $Q(\lambda,\beta,\varepsilon)$, $K(\lambda,\beta)$ agree satisfactorily with *P, I Q, K* in the sense of Points 1, 2, 3, 6 of the previous section.

In the following, we suppose that $\mathscr{P}$ is preferentially consistent (Assumption 1). Based on this, we shall reduce Problem 1 to a biobjective optimization problem, regardless of how many objectives compose the original formulation in (1).

4. A biobjective characterization of a best compromise solution

The next three definitions have been adapted from [10]:

Definition 4: Let *C* be a subset of *O*. If there does not exist $\boldsymbol{y} \in C$ such that $\boldsymbol{y}P(\lambda,\beta)\boldsymbol{x}$, we say that $\boldsymbol{x}$ is a $\mathscr{P}$–non strictly outranked solution in *C.*

Definition 5: $P(\lambda,\beta)$ is said to be free of inconsistencies iff there are no cycles of that relation in *O.*

Definition 6: $P(\lambda,\beta)$ is said to be minimally free of inconsistencies on *O* iff there does exist at least one $\mathscr{P}$-non-strictly outranked solution in *O.*

Definition 7: Let C be a subset of O. For each $\boldsymbol{x}$ in C, let us define the set of $\mathscr{P}$-strictly outranking solutions $(S_C)_x = \{\boldsymbol{y} \in C$ such that $\boldsymbol{y}P(\lambda,\beta)\boldsymbol{x}\}$. *card* $(S_C)_x$ is its cardinal, an integer function depending on $\boldsymbol{x}$. Obviously, if $\boldsymbol{x}$ is a $\mathscr{P}$-non-strictly outranked solution in O then *card* $(S_O)_x = 0$.

The following result has been adapted from [10]:

Proposition 1: The set of $\mathscr{P}$-non strictly outranked solutions in O is a subset of the Pareto frontier.

The proof is very simple. If the set of $\mathscr{P}$-non-outranked solutions in O is empty, it is a proper subset of the Pareto frontier. Otherwise, we should prove that

$\boldsymbol{a}$ is a $\mathscr{P}$-non strictly outranked solution in $O \Rightarrow \boldsymbol{a}$ is a Pareto solution.

Suppose that $\boldsymbol{a}$ is dominated by $\boldsymbol{b} \in O$. By definition of $P(\lambda,\beta)$ we have $\boldsymbol{b}P(\lambda,\beta)\boldsymbol{a}$. Hence, $\boldsymbol{b}$ $\mathscr{P}$-strictly outranks $\boldsymbol{a}$ in contradiction with the hypothesis.

The reciprocal of the above proposition is false. $\boldsymbol{a}$ may be a Pareto solution while being $\mathscr{P}$-strictly outranked by $\boldsymbol{b}$**,** simultaneously. It suffices to find $\boldsymbol{b}$ such that $\boldsymbol{b}P(\lambda,\beta)\boldsymbol{a}$ by satisfying Point *ii* or *iii* in the above definition of $P(\lambda,\beta)$ (Section 3). In such cases the set of $\mathscr{P}$-non strictly outranked solutions is a proper subset of the Pareto frontier.

Definition 8: The set $N_S = \{\boldsymbol{x} \in O$ such that *card* $(S_O)_x = 0\}$will be called the $\mathscr{P}$-non strictly outranked frontier of Problem 1.

Note that according to Def. 6, if $P(\lambda,\beta)$ is minimally free of inconsistencies the set N_S is not empty. This is empty only if every pair $(\boldsymbol{x},\boldsymbol{y}) \in O \times O$ is in some cycle of $P(\lambda,\beta)$.
The following proposition is trivial:

Proposition 2: Under Assumption 1, a best compromise solution for Problem 1 is some $\boldsymbol{x^*} \in N_S$.

Proof:
Suppose that $\boldsymbol{x^*}$is a best compromise and $\boldsymbol{x^*} \notin N_S$. Then there is $\boldsymbol{y} \in O$ such that $\boldsymbol{y}P(\lambda,\beta)\boldsymbol{x^*}$. Since $\mathscr{P}$ is preferentially consistent, $\boldsymbol{y}$ is strictly preferred to $\boldsymbol{x^*}$. This is a contradiction with our characterization of a best compromise given by Definition 2.

Definition 9: Let C be a subset of O. For each $\boldsymbol{x}$ in C, let us define the set of $\mathscr{P}$-weakly outranking solutions $(W_C)_x = \{\boldsymbol{y} \in C$ such that $\boldsymbol{y}Q(\lambda,\beta,\varepsilon)\boldsymbol{x}$ or $\boldsymbol{y}K(\lambda,\beta)\boldsymbol{x}$ $\}$. *card* $(W_C)_x$ is its cardinal, an integer function depending on $\boldsymbol{x}$.

Definition 10: Let C be a subset of O. $\boldsymbol{x} \in C$ is a $\mathscr{P}$-non weakly outranked solution in C iff *card* $(S_C)_x$ = *card* $(W_C)_x = 0$.

Definition 11: Let $A_P(\lambda,\beta,\varepsilon)$ be equal to $P(\lambda,\beta)\cup Q(\lambda,\beta,\varepsilon)\cup K(\lambda,\beta)$. $A_P(\lambda,\beta,\varepsilon)$ is said to be minimally free of inconsistencies on a set $C \subseteq O$ iff there does exist at least one $\mathscr{P}$-non weakly outranked solution in C.

Proposition 3: Under Assumption 1, $\boldsymbol{x^*}$is a best compromise solution for Problem 1 iff $\boldsymbol{x^*}$ is an ideal (0,0)-solution of the problem

$$\textbf{Minimize } (card(S_O)_{\boldsymbol{x}},\ card\ (W_O)_{\boldsymbol{x}}) \qquad (2)$$
$$\boldsymbol{x} \in O$$

Proof:

Since $\boldsymbol{x^*}$is a best compromise solution for Problem 1, there is no $\boldsymbol{y}\in O$ such that $\boldsymbol{y}A_P(\lambda,\beta,\varepsilon)\boldsymbol{x^*}$. Hence, $card(S_O)_{\boldsymbol{x^*}} = card\ (W_O)_{\boldsymbol{x^*}} = 0$.

On the other hand

$card(S_O)_{\boldsymbol{x^*}} = 0 \Rightarrow$ there is no $\boldsymbol{y}\in O$ such that $\boldsymbol{y}P(\lambda,\beta)\boldsymbol{x^*}$.

$card(W_O)_{\boldsymbol{x^*}} = 0 \Rightarrow$ there is no $\boldsymbol{y}\in O$ such that $\boldsymbol{y}Q(\lambda,\beta,\varepsilon)\boldsymbol{x^*}$.

$card(W_O)_{\boldsymbol{x^*}} = 0 \Rightarrow$ there is no $\boldsymbol{y}\in O$ such that $\boldsymbol{y}K(\lambda,\beta)\boldsymbol{x^*}$.

Under Assumption 1 and according to Definition 2, $\boldsymbol{x^*}$is a best compromise solution for Problem 1.

Remarks I:

1. Although its proof was trivial, Proposition 3 is an interesting result. Accepting Assumption 1, Problem 1 is transformed into a biobjective problem, regardless of the dimension of the original objective space. If a best compromise solution exists (in the sense of Definition 2) it should be the ideal solution for Problem 2. Besides, each point $\boldsymbol{x^*}\in O$ with $card(S_O)_{\boldsymbol{x^*}} = card\ (W_O)_{\boldsymbol{x^*}} = 0$ might be chosen as the final solution for Problem 1.
2. Under Assumption 1, there is a best compromise for Problem 1 iff $A_P(\lambda,\beta,\varepsilon)$ is minimally free of inconsistencies on O.
3. Suppose that (0,0) is not a solution for Problem 2. This implies an incorrect assessment of $A_P(\lambda,\beta,\varepsilon)$. The *DM* has two choices: i) To select the "best" non-dominated solution for Problem 2 as the final solution for his/her original problem, or ii) To modify the parameter settlement of $A_P(\lambda,\beta,\varepsilon)$.
4. Any multiobjective evolutionary algorithm may be used to solve Problem 2.

Def. 10 may be used for distinguishing a best compromise in N_S. Let us fix our attention on this set. Let us define the set of $\mathscr{P}$-non weakly outranked solutions in N_S, that is $N_W = \{\boldsymbol{y} \in N_S$ such that $card\ (W_{N_S})_{\boldsymbol{y}} = 0\}$.

Proposition 4: Under Assumption 1, a best compromise solution of (1) is some $\boldsymbol{x^*} \in N_W$.

Proof:

Suppose that $\boldsymbol{x^*}$is a best compromise: Hence N_S and N_W are not empty (Remark 2).

$\boldsymbol{x^*} \in N_S$ from Proposition 2 and suppose also that $\boldsymbol{x^*} \notin N_W$. Then, there is $\boldsymbol{y} \in N_S$ such that $\boldsymbol{y}A_P(\lambda,\beta,\varepsilon)\boldsymbol{x^*}$. This is a contradiction with the characterization of a best compromise given by Def. 2.

Proposition 5: Under Assumption 1, if $\boldsymbol{x^*}$is a best compromise solution for Problem 1, $\boldsymbol{x^*}$ is an ideal (0,0)-solution for the problem:

$$\textbf{Minimize } (card(S_O)_x,\ card\ (W_{NS})_x) \qquad (3)$$
$$\boldsymbol{x} \in O$$

Proof:
Since $\boldsymbol{x^*}$is a best compromise solution for Problem 1, from Proposition 3 $card(S_O)_{\boldsymbol{x^*}} = card\ (W_O)_{\boldsymbol{x^*}} = 0$. Obviously, if $C \subset O$ then $card\ (W_C)_{\boldsymbol{x}} \leq card\ (W_O)_{\boldsymbol{x}}$. Hence, $card\ (W_{NS})_{\boldsymbol{x^*}}=0$.

As a consequence from Proposition 5, a best compromise solution can be found through a lexicographic search, with preemptive priority favoring $card(S_o)_{\boldsymbol{x}}$. It is consistent with the fact that $P(\lambda,\beta)$ models the asymmetric preference information better than $Q(\lambda,\beta,\varepsilon)$ or $K(\lambda,\beta)$. This becomes more important when Assumption 1 is in question. In practical situations the *DM* may not be confident with the fuzzy outranking model given by $\sigma(\boldsymbol{x},\boldsymbol{y})$ or with a specific settlement of the model's parameters, including $(\lambda,\beta,\varepsilon)$. Changes in these parameters may modify the *DM*'s belief about the adequacy of Assumption 1. On this background, it is convenient to introduce the following definition:

Definition 12: $\boldsymbol{x^*}$ is a $\mathscr{P}$-best compromise solution in $C \subseteq O$ iff there is no $\boldsymbol{y} \in C$ such that $\boldsymbol{y}A_P(\lambda,\beta,\varepsilon)\boldsymbol{x^*}$.

Remarks II:

1. Propositions 2-5 remain valid when the notion of $\mathscr{P}$-best compromise is used instead of the previous concept given by Definition 2.
2. (0,0)-ideal solutions for Problems 2-3 are $\mathscr{P}$-best compromise ones. They may approximate best compromise solutions for Problem 1 as much as $A_P(\lambda,\beta,\varepsilon)$ is close to the actual asymmetric *DM*'s preference.
3. In practical situations the decision maker supported by a potential decision analyst should assess the set of model´s parameters included in $A_P(\lambda,\beta,\varepsilon)$ and σ. This is not an easy task, since decision-makers usually have difficulties in specifying outranking parameters and require an intense support by a decision analyst. To facilitate this process, the pair *DM*-decision analyst can use the preference disaggregation-analysis (*PDA*) paradigm (cf. [17]), which has received an increasing interest from the multicriteria decision support community. *PDA* infers the model's parameters from holistic judgments provided by the *DM*. Those judgments may be obtained from different sources (past decisions, decisions made for a limited set of fictitious objects (actions), or decisions taken for a subset of the objects under consideration for which the *DM* can easily make a judgment (cf. [8]).

In the framework of outranking methods *PDA* has been recently approached by [9, 11].

4. The non-existence of a $\mathscr{P}$-best compromise when using $A_P(\lambda,\beta,\varepsilon)$ does not imply the non existence of a best compromise solution according to Definition 2. A best compromise may exist under other asymmetric preference relations. Hence, the algorithmic search should be able to generate solutions even if $A_P(\lambda,\beta,\varepsilon)$ is not minimally free of inconsistencies.

Proposition 6: Suppose that the set of $\mathscr{P}$-best compromise solutions for Problem 1 is not empty. Under Assumption 1, $\boldsymbol{x^*}$ should be chosen as the best solution for Problem 1 only if it is a $\mathscr{P}$-best compromise solution.

This proposition can be justified as follows: If $\boldsymbol{x^*}$ is not a $\mathscr{P}$-best compromise solution, then there is a $\boldsymbol{y}$ such that $\boldsymbol{yA_Px^*}$. Under Assumption 1, the *DM* should feel certain preference favoring $\boldsymbol{y}$, thus questioning $\boldsymbol{x^*}$ as the best solution**.**

Remarks III:

1. In practical situations, the value of the information contained in $A_P(\lambda,\beta,\varepsilon)$ is a central issue. The *DM* should be confident with respect to $P(\lambda,\beta)$. This means that a strictly outranked solution should not be the final solution for Problem 1. Such remark allows "filtering" the *MOP*'s Pareto frontier, thus obtaining a privileged portion of the nondominated set. However, a normative use of $Q(\lambda,\beta,\varepsilon)$and $K(\lambda,\beta)$ is more questionable due to i) the cognitive limitations of real decision-makers, and ii) some model imprecisions. Hence, if $\boldsymbol{x'} \in N_S$, then $\boldsymbol{y}A_P(\lambda,\beta,\varepsilon)\boldsymbol{x'}$ may not be a sufficiently strong argument to discard $\boldsymbol{x'}$. More information could be needed to perform a reliable comparison.
2. $A_P(\lambda,\beta,\varepsilon)$ is created by comparing pairs of actions with respect to the set of criterion functions. Therefore $\sigma(\boldsymbol{x},\boldsymbol{y})$ only contains information about the vectors $\boldsymbol{F}(\boldsymbol{x})$ and $\boldsymbol{F}(\boldsymbol{y})$. If $card\ (N_W) > 1$, then there is no reason based on $A_P(\lambda,\beta,\varepsilon)$ which allows us to distinguish the best final solution. The *DM* confidence on the $\mathscr{P}$-best compromises may be enhanced by using additional information contained in the fuzzy outranking relation, namely some quality measure obtained from a comparison of each particular solution with the remaining ones which are potential candidates to be the final solution.
3. When $card\ (N_W)= 0$ being $card\ (N_S)\neq 0$, there is not a $\mathscr{P}$-best compromise solution for Problem 1. The information contained in $K \cup Q$ is not consistent. The *DM* should use the above discussed additional information in order to select the final solution from N_S.

Here, we suggest to use the outranking net flow score. This is a very popular measure to rank a set of alternatives on which a fuzzy preference relation is defined (cf. [13]). If $\sigma(x,y)$ is a fuzzy preference relation on a set A, the net flow score associated to $a \in A$ is defined as $F_n(a) = \ \Sigma_{c\in A-\{a\}}\ [\sigma(a,c) - \sigma(c,a)]$. Note that $F_n(a) > F_n(b)$is an asymmetric and transitive binary relation on C, also indicating some kind of preference on this set. So, the net flow score may be used to select the most satisfactory solution when the *DM* is not sufficiently confident on $A_P(\lambda,\beta,\varepsilon)$. Guimaraes, Massebeuf et al. and Fernandez et al. ([12, 14, 19]) have used the net flow score in the context of multiobjective evolutionary algorithms.

Definition 13: Let C be a subset of O and $F_n(\boldsymbol{a})$, the net flow score associated to $\boldsymbol{a} \in C$. For each $\boldsymbol{x}$ in C, let us define the set of net flow outranking solutions $(F_C)_x = \{\boldsymbol{y} \in C$ such that $F_n(\boldsymbol{y}) > F_n(\boldsymbol{x})\}$. $card\ (F_C)_x$ is its cardinal, an integer function depending on $\boldsymbol{x}$.

Taking into account the net flow score information, we propose to select the best solution for Problem 1 from the nondominated set obtained from:

$$\textbf{Minimize } (card(S_O)_{\boldsymbol{x}},\ card\ (W_{Ns})_{\boldsymbol{x}},\ card(F_{Ns})_{\boldsymbol{x}}) \quad (4)$$
$$\boldsymbol{x} \in O$$

Note that the preemptive priority favoring $card(S_o)$ is kept by Problem 4.

Generally, N_S is not empty. $P(\lambda,\beta)$ is not minimally free of inconsistencies on O (usually a very large set) only if each pair of $O \times O$ is on a cycle of that strict preference relation. This condition is very hard with high values of λ and β. Nevertheless, when N_S is empty, the decision maker can set higher values for those parameters in order to break some cycles of $P(\lambda,\beta)$, thus making this relation minimally free of inconsistencies.

As a generalization of (4), this approach works even if the asymmetric preference relations do not hold the minima consistency property. We suggest to solve the problem:

$$\textbf{Minimize } (card(S_o)_{\boldsymbol{x}},\ card\ (W_B)_{\boldsymbol{x}},\ card(F_B)_{\boldsymbol{x}}) \quad (5)$$
$$\boldsymbol{x} \in O$$

with preemptive priority favoring $card(S_o)$. B is the set $\{\boldsymbol{y} \in O$ such that $\boldsymbol{y} = arg\ min\ card(S_o)_{\boldsymbol{x}}\}$. Since $card(S_o)_{\boldsymbol{x}}$ is bounded B is always not empty. $B = N_S$ when $P(\lambda,\beta)$ is minimally free of inconsistencies. When $Q(\lambda,\beta,\varepsilon) \cup K(\lambda,\beta)$ is also minimally free of inconsistencies on N_S, a $\mathscr{P}$-best compromise solution may be obtained by solving (5) giving preemptive priority to $card\ (W_B)_{\boldsymbol{x}}$.

5. Adapting the *NSGA-II* to solve Problem 5: the extended *NOSGA* method

For solving (4-5) we propose a method inspired on the *NSGA-II* (cf. [6]), but a) making selective pressure toward the $\mathscr{P}$-non-strictly outranked frontier, and b) looking for nondominated solutions obtained from minimizing ($card\ (W_B)_{\boldsymbol{x}}$, $card(F_B)_{\boldsymbol{x}}$). In fact, the antecedent of this method was called *NOSGA* by [10], acronym from Non-Outranked-Sorting Genetic Algorithm. Its main idea is similar to *NSGA-II*, but working with $\mathscr{P}$-non strictly outranked individuals instead of nondominated ones. The “filtering” process is similar, but extracting $\mathscr{P}$-non strictly outranked individuals which form classes with the same value of $card\ (S_O)$. Such process guarantees the lexicographic priority of the first objective in Problems 4-5. As in the *NSGA-II*, when in binary tournaments $card\ (S_O)_{\boldsymbol{x}} < card\ (S_O)_{\boldsymbol{y}}$ the first individual is chosen. Unlike typical *MOEA*s, we are not interested in obtaining a uniform distribution of solutions representing the Pareto frontier. Therefore, instead of the *NSGA-II*'s crowding distance (or another density estimator), for each $\boldsymbol{x}$ in the current population, we propose to use a merit measure defined as $\eta = card\ (W_{front})_{\boldsymbol{x}} + card(F_{front})_{\boldsymbol{x}}$ (*front* is the set composed by $\boldsymbol{x}$ and the individuals with the same $card\ (S_O)_{\boldsymbol{x}}$. That is, when two individual with equal $card(S_O)$ are compared (in binary tournaments or deciding who will be included into the new generation), the least η will be preferred. The goal is to increase the selective pressure towards most satisfactory solutions looking for $\mathscr{P}$-non strictly outranked solutions with good values of $card\ (W_{NS})$ and $card(F_{NS})$. The pseudocode of this extended *NOSGA* method is shown below:

```
PROCEDURE NOSGA-II (L, Number_of_Generations)
Initialize Population P
Generate random population with size L
Evaluate objective values
Evaluate σ on P×P
For each x ∈ P, calculate card (S_O)_x
Generate fronts of equal values of card (S_O )
Assign to these fronts a rank (level) based on card (S_O )
 FOR each rank C_i  DO
        for each x∈ C_i, calculate card (W_Ci)_x, card(F_Ci)_x, η
 End FOR
Generate Child Population Q with size L
        Perform Binary Tournament Selection
        Perform Recombination and mutation
FOR I = 1 to Number_of_Generations  DO
      Assign P' = P ∪ Q
      Evaluate σ on P'×P'
      FOR each parent and child in P' DO
          Calculate card (S_O)
          Assign rank (level) based on card (S_O)
          Calculate η
          Loop (inside) by adding solutions to the
          next generation until L individuals have been found
     End FOR
     Replace P by the L individuals found
     Generate Child Population Q with size L
           Perform Binary Tournament Selection
           Perform Recombination and mutation
End FOR
End PROCEDURE
```

In comparison with *NOSGA* the main differences are: a) Finding the non strictly outranked frontier, *NOSGA* solves a single-criterion optimization problem; b) *NOSGA-II* is based on an enhanced theoretical characterization of most satisfactory solutions for *MOP*s; c) the use of $Q(\lambda,\beta,\varepsilon)$and $K(\lambda,\beta)$ in *NOSGA-II*; d) as a consequence of points b) and c), *NOSGA-II* promises an increase of the selective pressure toward the best portion of the Pareto frontier.

6. An illustrative example

Let us consider a decision making situation in which the *DM* is choosing among *L'* different public policies (projects) each with a direct social impact. This is measured by using a nine-component vector (N_1, N_2,…. N_9). $N_i = n_{kj}$ denotes the number of people belonging to the *k*-th social category which receive the *j*-th benefit level from that policy or project. In this example *k*= 1, 2, 3 correspond to (Extreme Poverty, Poverty, Middle), and *j* = 1, 2, 3 to (High Impact, Middle Impact, Low Impact). N_1, N_2, N_3 correspond to extreme poverty people; N_7, N_8, N_9 concern middle class. For example, consider the vector (1000, 0,0, 2000, 0,0,0, 1500, 0) associated to a particular project. This means that 1000 extreme-poverty people and 2000 poor people receive high-impact benefits from the project; besides, 1500 middle-class people receive middle-impact benefits from the same project. Note that these quantities can be added when a set of projects is considered. In the following N_i^m denotes the value of N_i associated to the *m*-th project. *C'* denotes a portfolio (a subset of the *L'* projects which receives financial support). The value of N_i for the whole portfolio is $N_i\ (C') = x_1\ N_i^1 + …. + x_L\ N_i^{L'}$ where $x_j = 1$ if the *j*-th project is supported and $x_j = 0$, otherwise. The set of applicant projects is composed of 100 proposals. The aim of this decision problem is to choose the "best" portfolio satisfying some budget constraints. Formally, the problem is:

$$\underset{C' \in R_F}{Maximize}\ (N_1(C'), N_2(C'),…. N_9(C')) \qquad (6)$$

where R_F is a feasible region determined by budget constraints.

Budget constraints are imposed by the class of project (educational, health, etc.), geographic region and to the whole portfolio. The total available budget was set as Total_budget= 2.5 billion dollars. The constraints by class and region are given by:

$$\begin{array}{l} 0.3 \text{ Total\_budget} \le \text{Budget\_Class } 1 \le 0.4 \text{ Total\_budget} \\ 0.25 \text{ Total\_budget} \le \text{Budget\_Class } 2 \le 0.35 \text{ Total\_budget} \\ 0.2 \text{ Total\_budget} \le \text{Budget\_Class } 3 \le 0.3 \text{ Total\_budget} \\ 0.4 \text{ Total\_budget} \le \text{Budget\_Region } 1 \le 0.6 \text{ Total\_budget} \\ 0.4 \text{ Total\_budget} \le \text{Budget\_Region } 2 \le 0.6 \text{ Total\_budget} \end{array} \qquad (7)$$

The degree of truth $\sigma(\boldsymbol{x},\boldsymbol{y})$ of the statement "$\boldsymbol{x}$ is at least as good as $\boldsymbol{y}$" is calculated as in ELECTRE-III method, that is:

$$\sigma(\boldsymbol{x},\boldsymbol{y}) = c(\boldsymbol{x},\boldsymbol{y}).\ N(d(\boldsymbol{x},\boldsymbol{y})) \qquad (8)$$

where:

$c(\boldsymbol{x},\boldsymbol{y})$ denotes the degree of truth of the concordance predicate;

$N(d(\boldsymbol{x},\boldsymbol{y}))$ denotes the degree of truth of the non-discordance predicate.

We shall take

$$c(\boldsymbol{x},\boldsymbol{y}) = \sum_{j \in C_{x,y}} w_{\mathrm{j}} \qquad (9)$$

where $C_{x,y}$ is the concordance coalition and *w*'s denote "weights" ($w_1 + w_2 + ... + w_n = 1$).

Let $D_{x,y}$ be the discordance coalition with ***xSy***. The intensity of discordance is measured in comparison with a veto threshold v_j, which is the maximum difference y_j-x_j compatible with $\sigma(\boldsymbol{x},\boldsymbol{y})>0$. As in [10] we shall use here:

$$N(d(\boldsymbol{x},\boldsymbol{y})) = \min_{j\in D_{x,y}} [1 - d_j(\boldsymbol{x},\boldsymbol{y})] \tag{10}$$

$$d_j(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 1 & \text{iff } \nabla_j \geq v_j \\ (\nabla_j - u_j)/(v_j - u_j) & \text{iff } u_j < \nabla_j < v_j \\ 0 & \text{iff } \nabla_j \leq u_j \end{cases} \tag{11}$$

where $\nabla_j = y_j$-x_j and u_j is a discordance threshold (see Figure 1).

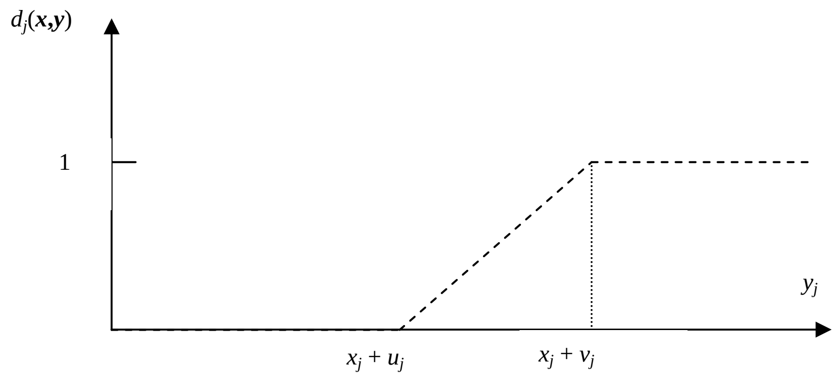

Figure 1 Partial discordance relation $d_j(\boldsymbol{x},\boldsymbol{y})$

We use binary encoding; a '1' in the individual *j*-th allele means that the *j*-th project belongs to this particular portfolio. Other parameters of the evolutionary search are: crossover probability = 1; mutation probability = 0.02; population size = 100.

Preference model parameters:
Taking into account the importance of each objective, the normalized weights were settled by the decision-maker as (0.23, 0.14, 0.11, 0.14, 0.11, 0.07, 0.09, 0.07, 0.04). The indifference thresholds were calculated as a measure of the error evaluating each objective, which was assessed as the 15% of its maximum value on the set of projects. The veto thresholds were settled as 0.5*(Max f_i - Min f_i) as in some applications of ELECTRE methods (cf. [21]); operations Max and Min act on a population. $P(\lambda,\beta)$ and $Q(\lambda,\beta,\varepsilon)$ were obtained by setting $\lambda= 0.67$, $\beta= 0.2$, $\varepsilon=0.1$.
We experienced with five random instances. Coded in Microsoft Visual C# 2005 (framework 2.0), the average run time was 5.1 minutes on a laptop computer with 1.66 GHz Intel® Core™ 2 Duo microprocessor, 2 GB RAM 667MHz DDR2, 120GB hard disk, and Windows® Vista Business.

The (known) $\mathscr{P}$–non outranked frontier of one random instance of this problem is shown in Table 1. The objective values are given in thousands.

Table 1: Approximation to the $\mathscr{P}$–non strictly outranked frontier

Portfolio	**N_1**	**N_2**	**N_3**	**N_4**	**N_5**	**N_6**	**N_7**	**N_8**	**N_9**	**Card(W)**	**F_n**
1	820	620	675	1,095	1,005	810	1,086	660	630	0	1.81
2	820	645	655	1,155	1,005	780	990	636	726	2	1.32
3	820	685	610	1,095	1,005	780	1,050	636	672	7	0.99
4	820	665	625	1,155	1,020	780	990	636	720	2	0.95
5	820	690	610	1,155	960	780	1,026	642	720	4	0.91
6	820	690	605	1,155	960	780	1,050	600	720	4	0.85
7	820	620	655	1,155	1,035	750	1,014	636	714	2	0.47
8	820	690	625	1,155	990	780	990	642	720	4	0.46
9	820	695	590	1,155	1,035	735	984	636	720	3	0.16
10	820	730	555	1,095	1,080	690	1,044	624	672	3	0.13
11	820	630	655	1,155	1,020	750	1,014	636	714	0	0.05
12	820	710	625	1,095	1,005	780	1,014	642	672	5	0.03
13	820	615	675	1,095	960	840	1,086	660	582	0	0.00
14	820	665	610	1,155	990	780	1,026	636	720	5	0.27
15	820	675	635	1,155	975	780	990	636	726	4	0.49
Ideal	820	740	830	1,260	1,095	1,065	1,104	696	870		
Nadir	300	335	420	570	570	510	540	264	438		

Card(W) and F_n are calculated on N_S

Portfolios 1, 11 and 13 hold the necessary conditions to be $\mathscr{P}$-best compromise solutions on *O*. Portfolio 1 is the single nondominated solution for Problem 4. If the *DM* were not confident with respect to $Q \cup K$, he/she might consider also Solution 2.

Those 15 solutions were compared with ten runs of *NOSGA*. In the following $N1_k$ denotes the set of non strictly outranked solutions obtained by *NOSGA* in its *kth* run. *N2* is the set shown in Table 1. *U* will denote the union set of $N1_k$ and *N2*, and NO_U the non-strictly outranked set in *U*. Some results are pointed-out in Tables 2-3.

Table 2. Net flow score calculated on *U* (NOSGA versus NOSGA-II)

Set	Average	Minimum	Maximum
$N1_1$	-2.9	-3.3	-2.3
N2	0.5	-0.4	1.2
$N1_2$	-2.1	-4.1	-0.5
N2	0.8	-0.1	1.6
$N1_3$	-2.2	-4.9	0.0
N2	2.2	0.3	4.7

Set	Average	Minimum	Maximum
$N1_4$	-3.7	-5.0	-0.9
N2	2.0	0.1	3.0
$N1_5$	-2.0	-4.5	0.0
N2	2.0	-1.5	4.8
$N1_6$	-3.4	-5.8	1.0
N2	3.9	0.8	5.6
$N1_7$	-2.4	-4.7	-0.5
N2	1.2	0.0	2.1
$N1_8$	-1.0	-2.9	1.4
N2	0.5	-0.8	2.6
$N1_9$	-2.7	-4.7	-0.6
N2	0.5	-0.8	1.7
$N1_{10}$	-2.6	-7.1	0.1
N2	3.1	0.4	5.5

Table 3. Robustness of $N1_k$ and N2 (NOSGA versus NOSGA-II)

Run	Card ($N1_k$)	Card ($N1_k \cap NO_U$)	Card ($N2 \cap NO_U$)
1	3	0	15
2	6	0	15
3	15	0	15
4	8	1	15
5	15	3	15
6	17	1	15
7	8	2	15
8	8	1	10
9	3	0	15
10	18	0	13

The following remarks come from Tables 2-3:
First: The net flow score of solutions by *NOSGA* is very low compared with those obtained by *NOSGA-II*. Eight times to ten the worst solution in N2 outperforms the best solution in $N1_k$.
Second: Only three solutions by *NOSGA* remain non-strictly outranked in *U* (third column of Table 3, fifth run). In comparison, ten solutions coming from a single *NOSGA-II* run (Table 1) are non-strictly outranked in that union set.
An approximation to the Pareto front was obtained for the same five instances using the standard *NSGA-II*, which is still the benchmark in evolutionary multiobjective optimization (e.g. [1, 27, 28]). In the following, NO_k and ND_k will denote the best front obtained by *NOSGA-II* and *NSGA-II*, respectively, for the *k*-th instance. Let *U* be $NO_k \cup ND_k$. Let NO_U and ND_U be the non-strictly outranked set and the nondominated set in *U*, respectively. A comparison between NO_k and ND_k was performed in such five random instances with the results shown in Tables 4, 5 and 6:

Table 4. Net flow score calculated on U (NOSGA-II versus NSGA-II)

Set	Average	Maximum value	Minimum value

Set	Average	Maximum value	Minimum value
NO_1	37.71	42.35	30.29
ND_1	-7.92	8.36	-34.49
NO_2	45.31	57.56	31.28
ND_2	-13.14	37.66	-52.46
NO_3	50.24	62.85	45.55
ND_3	-17.35	19.01	-65.85
NO_4	60.13	62.08	56.55
ND_4	-9.02	27.79	-41.00
NO_5	63.58	66.74	55.53
ND_5	-11.44	22.63	-45.53

Table 5. Robustness of NO_k (NOSGA-II versus NSGA-II)

Instance	Card(NO_k)	Card(NO_U)	Card($NO_k \cap NO_U$)	Card($NO_k \cap ND_U$)
1	21	25	21	21
2	29	34	29	29
3	32	32	32	32
4	15	15	15	15
5	18	18	18	18

Table 6. Robustness of ND_k

Instance	Card(ND_k)	Card(NO_U)	Card($ND_k \cap NO_U$)	Card($ND_k \cap ND_U$)
1	100	25	5	66
2	100	34	5	88
3	100	32	0	82
4	100	15	0	77
5	100	18	0	82

From Tables 4-6, it should be noticed that

1. The net flow score of solutions in ND_k is very low compared with those in NO_k;
2. Each $\boldsymbol{x} \in NO_k$ is not dominated in U (fifth column of Table 5);
3. Each $\boldsymbol{x} \in NO_k$ remains as non-strictly outranked in U (fourth column of Table 5);
4. Only a few non-strictly outranked solutions are added by ND_k (in two instances, first two rows, fourth column of Table 6). However, those solutions are outperformed by others belonging to NO_k;
5. In three instances, no $\boldsymbol{x} \in ND_k$ is member of NO_U; *NSGA-II* does not find the non-strictly outranked set;
6. The differences between $card(ND_k)$ and $card(ND_k \cap ND_U)$ indicate that 12-34% of the solutions belonging to ND_k are actually dominated by some element of NO_k.

In Table 7, the first column shows the cardinal of the known $\mathscr{P}$–best compromise solutions in N_S. The cardinal of the known Pareto frontier of Problem 4 (denoted by *PF4*) is pointed out in the second column.

Table 7. Privileged $\mathscr{P}$–non outranked solutions

Instance	Card (NW)	Card (PF4)	Card (NW $\cap$PF4)
1	2	2	1
2	1	6	1
3	0	4	0
4	3	1	1
5	0	3	0

According to the discussion in Section 4, under Assumption 1, the best solution should be an element of $N_W \cap PF4$. Otherwise, the *DM* should consider the elements in *PF4*. These sets contain the non-strictly outranked solutions with highest net flow score and lowest "weakness". In instances 3 and 5 the solution with highest net flow score is not necessarily the best one. It depends on 1) the comparison of *card(W)* and F_n , and 2) how confident the *DM* is on *Q* and *K*.

7. Conclusions

A relational system of preferences coming from multicriteria decision aid, which includes strict preference, weak preference, *K*-preference, indifference and incomparability binary relations, is useful to characterize a best compromise solution of a multiobjective optimization problem. Such characterization seems to be the most interesting result of this paper. A fuzzy outranking relation can be used to build a model of the *DM*'s asymmetric preference relation, which is a central issue in approaching most satisfactory solutions.
Under certain conditions of a decision maker's consistency, a best compromise solution is obtained as the ideal solution of a bi-objective optimization problem, which is a map of the original problem. When the *DM* is confident on the asymmetric preference model, he/she should accept as satisfactory the ideal solution for that bi-objective problem. Otherwise, a best compromise solution may be obtained from a three-objective problem, in which net-flow score information is incorporated for improving the asymmetric preference model.
The above characterization is very useful in order to achieve selective pressure towards preferential privileged Pareto solutions of the original problem. Using it, the *NOSGA-II* algorithm, which is a derivation from the *NSGA-II* for solving the lexicographic mapped formulation, seems capable of obtaining very good approximations to the best compromise. *NSGA-II* and the original *NOSGA* method are clearly outperformed in several random instances of a real-size problem.
It is also interesting that the equivalence between the original *MOP* and its mapped two or three-objective problem is valid independently of the original objective space dimension. This may be very important to solve *MOP*s with many objective functions.

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