

# An Overview of Weighted and Unconstrained Scalarizing Functions

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**Abstract.** Scalarizing functions play a crucial role in multi-objective evolutionary algorithms (MOEAs) based on decomposition and the R2 indicator, since they guide the population towards nearly optimal solutions, assigning a fitness value to an individual according to a predefined target direction in objective space. This paper presents a general review of weighted scalarizing functions without constraints, which have been proposed not only within evolutionary multi-objective optimization but also in the mathematical programming literature. We also investigate their scalability up to 10 objectives, using the test problems of Lamé Spheres on the MOEA/D and MOMBI-II frameworks. For this purpose, the best suited scalarizing functions and their model parameters are determined through the evolutionary calibrator EVOCA. Our experimental results reveal that some of these scalarizing functions are quite robust and suitable for handling many-objective optimization problems.

**Keywords:** scalarizing function, many-objective optimization, evolutionary algorithms, tuning process.

## 1 Introduction

Multi-Objective Evolutionary Algorithms (MOEAs) mostly rely on three methods for tackling Multi-objective Optimization Problems (MOPs): *Pareto dominance*<sup>4</sup>, aggregation, and performance indicators. MOEAs based on Pareto dominance compare individuals preferring those that are less dominated by other members in the population. When a tie occurs, a secondary selection criterion is applied to improve *diversity*, i.e., the uniform distribution of solutions covering all regions in objective space. Aggregation-based MOEAs decompose MOPs into several single-objective subproblems, each one associated with a different target direction or *weight vector*. The best individuals for every subproblem have a better chance to survive. Indicator-based MOEAs favor solutions that highly

<sup>4</sup> A solution  $\mathbf{x} \in \mathcal{S}$  dominates a solution  $\mathbf{y} \in \mathcal{S}$  ( $\mathbf{x} \prec \mathbf{y}$ ), if and only if  $\forall i \in \{1, \dots, m\}$ ,  $f_i(\mathbf{x}) \leq f_i(\mathbf{y})$  and  $\exists j \in \{1, \dots, m\}$ ,  $f_j(\mathbf{x}) < f_j(\mathbf{y})$ .

contribute to a performance indicator, which reflects a quality aspect regarding convergence and diversity of the current population. In this case, *convergence* measures how close the solutions are to the *Pareto Optimal Front (POF)*<sup>5</sup>. The hypervolume [1] and the unary *R2* [2] indicators are examples of quality measures that incorporate both aspects, and they are also Pareto compliant.

One of the main concerns is that Pareto-based MOEAs are ineffective in MOPs having four or more objective functions, the so-called many-objective optimization problems (MaOPs) [3]. Owing to the fact that at early generations, most individuals will normally become non-dominated, and the search will then be guided solely by the secondary selection criterion. In consequence, individuals that enhance diversity are kept although they may deteriorate convergence [4]. The hypervolume indicator does not suffer from this selection pressure issue since its maximization is equivalent to reaching the Pareto Optimal Set (POS) [5]. However, its computational cost increases exponentially with the number of objectives [6], making it unaffordable for MaOPs.

On the other hand, the methods based on aggregation and the *R2* indicator can scale to any number of objectives while having a low computational cost. Nevertheless, two key points should be considered when using such approaches: the setting of the scalarizing function and the weight vectors. A *scalarizing function*, also known as utility function or aggregation function, transforms the original MOP into a real value using a predefined weight vector in objective space. Regarding the choice of the weight vectors, several attempts have been proposed for adapting them in order to get a high-quality approximation of the Pareto front (see for example [7, 8]). However, not much is known about the proper selection of the scalarizing function and its effect on MOEAs. In fact, only three aggregation functions have been exhaustively researched so far [9–11], neglecting other approaches that may be able to handle MaOPs.

In this paper, we present a general review of fifteen weighted unconstrained scalarizing functions. Even though there are excellent reviews on this topic [12–15], the most comprehensive study dates from 1998 [13], while the most recent ones only consider a small set of utility functions. Furthermore, none of these studies analyzes their ability to scale to any number of objectives.

This paper is organized as follows. In Section 2 we present a review of unconstrained scalarizing functions. In Section 3, the best scalarizing functions and their model parameters are determined through the Evolutionary Calibrator (EVOCA) [16], using both the Multiobjective Evolutionary Algorithm Based on Decomposition (MOEA/D) [17] and the Many-Objective Metaheuristic Based on the *R2* Indicator II (MOMBI-II) [18]. Finally, our conclusions and some possible paths for future research are provided in Section 4.

## 2 Scalarizing Functions

We focus on unconstrained scalarizing functions that transform a multi-objective optimization problem (MOP) into a single-objective problem, using a predefined

<sup>5</sup>  $POF := \{\mathbf{F}(\mathbf{x}) \in \mathbb{R}^m : \mathbf{x} \in \mathcal{S}, \nexists \mathbf{y} \in \mathcal{S}, \mathbf{y} \prec \mathbf{x}\}$ .

weight vector  $\mathbf{w} := (w_1, \dots, w_m)$  of the following form:

$$\text{minimize } u(\mathbf{f}'(\mathbf{x}); \mathbf{w}) \quad (1)$$

$$\text{subject to } \mathbf{x} \in \mathcal{S}, \quad (2)$$

where  $\mathbf{x}$  is the decision vector,  $\mathcal{S} \in \mathbb{R}^n$  is the feasible space,  $\mathbf{f} \in \mathbb{R}^m$  is the vector of  $m(\geq 2)$  objective functions,  $\mathbf{f}'(\mathbf{x}) := \mathbf{f}(\mathbf{x}) - \mathbf{z}$  and  $\mathbf{z} := (z_1, \dots, z_m)^T$  is a reference point. Unless otherwise stated we used as reference point the ideal point, i.e.,  $z_i := \min \{f_i(\mathbf{x}) \mid \mathbf{x} \in \mathcal{S}\} \forall i \in \{1, \dots, m\}$ . Each component of  $\mathbf{w}$  must satisfy  $w_i \geq 0$ . Although there is no particular reason beyond achieving uniformity among solutions, we also assume that  $\sum_i w_i = 1$  [15].

In the remainder of this section, we present a review of fifteen aggregation functions. They differ in that they minimize some sort of a distance metric to the reference point, others combine two distance metrics, and a minority of them also consider the deviation to the weight vector. In all cases, their computational complexity is  $O(m)$ . Several of these scalarizing functions can generate (weakly) Pareto optimal solutions.<sup>6</sup> Moreover, interactive methods have coupled some of these scalarizing functions, where the reference point reflects the decision maker's preferences [13, p. 131], [19, 14, 20, 21]. Let  $p \in \mathbb{N}_+$ ,  $\alpha \in \mathbb{R}_+$  and  $\theta \in \mathbb{R}$  be the model parameters. The dot product is symbolized as  $\bullet$ , the absolute value of a real number is denoted by  $|\cdot|$ , and the magnitude of a vector is represented by  $\|\cdot\|$ .

The **Weighted Compromise Programming (WCP)** [22] is derived from the *Global Criterion Method* [23, p. 32], which includes the weight vector for modeling preferences as follows:

$$u^{wcp}(\mathbf{f}'; \mathbf{w}) := \sum_i (w_i f'_i)^p. \quad (3)$$

A high value of  $p \in (1, \infty)$  is preferred to obtain the complete POS [24]. In [25], the authors recommend using odd values for this parameter and coupled WCP ( $p = 9$ ) with a metaheuristic for solving convex and concave Pareto fronts with 2 and 3 objectives.

The **Weighted Sum (WS)** [26] is one of the most commonly used scalarizing functions, which linearly combines the objectives as follows:

$$u^{ws}(\mathbf{f}'; \mathbf{w}) := \sum_i w_i f'_i. \quad (4)$$

However, WS cannot generate solutions in concave regions of the Pareto front [25]. Some attempts have been proposed to alleviate this drawback, such as its combination with other scalarizing functions [27–29], the use of dynamic weights coupled with a secondary population [30], and the use of WS as a local search engine [31, 32]. Additionally, some studies have reported that WS is an effective method for solving MaOPs [9, 33]. This scalarizing function has been integrated into several evolutionary algorithms (see e.g., [17, 34]).

<sup>6</sup> Let be  $\mathbf{x}, \mathbf{y} \in \mathcal{S}$ . It is said that  $\mathbf{x}$  is Pareto optimal if there is no  $\mathbf{y}$  such that  $\mathbf{y} \prec \mathbf{x}$ .  $\mathbf{x}$  is weakly Pareto optimal if there is no  $\mathbf{y}$  such that  $\forall i \in \{1, \dots, m\}, f_i(\mathbf{y}) < f_i(\mathbf{x})$ .

The **Exponential Weighted Criteria (EWC)** [24] can deal with any Pareto-front shape, and is given by:

$$u^{ewc}(\mathbf{f}'; \mathbf{w}) := \sum_i (e^{w_i} - 1) e^{f'_i}. \quad (5)$$

A large value of  $p$  is required to achieve Pareto optimality, but this can lead to numerical overflow [12]. In [35], EWC was used to solve a problem related to a voltage distribution network. To the best of our knowledge, this scalarizing function has not been integrated into any MOEA until now.

The **Weighted Power (WPO)** [36] relies on the principle that the POF can be convexified as follows:

$$u^{wpo}(\mathbf{f}'; \mathbf{w}) := \sum_i w_i (f'_i)^p. \quad (6)$$

For a suitable value of  $p \in [1, \infty)$ , this scalarizing function can also find optimal solutions in concave Pareto fronts [12, 37]. In [38], WPO was coupled with a genetic algorithm, where the weight vectors and the exponent  $p$  were updated during the evolution process according to predefined rules.

The **Weighted Product (WPR)** [39, p. 9], also called *product of powers*, is defined as follows:

$$u^{wpr}(\mathbf{f}'; \mathbf{w}) := \prod_i (f'_i)^{w_i}. \quad (7)$$

This scalarizing function has been integrated with several ant colony optimization algorithms [40] and it has also been applied to solve a network design problem [41]. However, this approach has not been widely used with MOEAs.

The **Weighted Norm (WN)** [22], or *weighted  $L_p$ -metrics*, is given by:

$$u^{wn}(\mathbf{f}'; \mathbf{w}) := \left( \sum_i w_i |f'_i|^p \right)^{\frac{1}{p}}. \quad (8)$$

WN can generate Pareto optimal solutions when  $p \in [1, \infty)$  [13, p. 98]. Moreover, other scalarizing functions can be derived from it, such as WS with  $p = 1$  and Least Squares [13, p. 97],[42] with  $p = 2$ . Moreover, concave Pareto fronts can be covered with a larger value of  $p$ . In [43], the behavior of WN was studied on MOEA/D, using continuous test problems having up to 7 objectives. The value of  $p$  was adaptively fine-tuned based on a local estimation of the Pareto front shape, taking different values from the set  $\{1/2, 2/3, 1, 2, 3, \dots, 10, 1000\}$ .

The **Chebyshev or Tchebycheff function (TCH)**<sup>7</sup> [44], also known as the weighted min-max [28, 12], is a particular case of WN with  $p = \infty$ , given by:

$$u^{tch}(\mathbf{f}'; \mathbf{w}) := \max_i \left\{ w_i |f'_i| \right\}. \quad (9)$$

This scalarizing function can find at least weakly Pareto optimal solutions regardless of the Pareto-front shape (whenever  $\mathbf{w} \in \mathbb{R}_+^m$ ) [13, p. 98]. By definition,

<sup>7</sup> Although the French spelling Tchebycheff is the most preferred, the proper English transliteration is Chebyshev.

$f'_i$  is guaranteed to be always positive. Thus, the absolute value could be discarded. Even though it is possible to obtain all Pareto optimal solutions for some weight vector [13, p. 99], a recent analysis has revealed that the search ability of TCH is equivalent to Pareto-based methods [45]. Thus, in MaOPs, the probability to obtain non-dominated solutions using TCH is lower than the WN ( $0 < p < \infty$ ) and equivalent to Pareto-based methods. TCH has been applied to different MOEAs (see, e.g., [17, 46]).

The **Augmented Chebyshev (ATCH)** [19] is a variant of TCH, where an extra term is considered in order to avoid the generation of weakly Pareto optimal solutions. It is given by:

$$u^{atch}(\mathbf{f}'; \mathbf{w}) := \max_i \left\{ w_i |f'_i| \right\} + \alpha \sum_i |f'_i|. \quad (10)$$

A too small value of  $\alpha$  may result in a loss of significance of the additional term, still leading to the generation of weakly Pareto optimal solutions. However, a too large value of this parameter may cause that some non-dominated points become unreachable [47]. Although the recommendation is to use small values of  $\alpha$ , such as [0.001, 0.01] [19], some studies have shown a better performance when using large values, revealing a high sensitivity of this parameter on discrete MaOPs [27].

The **Modified Chebyshev (MTCH)** [48] is a variation of TCH, given by:

$$u^{mtch}(\mathbf{f}'; \mathbf{w}) := \max_i \left\{ w_i \left( |f'_i| + \alpha \sum_i |f'_i| \right) \right\}. \quad (11)$$

$\alpha$  should be a small positive value. The features of this method are discussed and illustrated in [13]. To the best of our knowledge, this scalarizing function has not been exploited in any MOEA.

The **Achievement Scalarizing Function (ASF)** [49] can produce weakly Pareto optimal solutions, expressed as:

$$u^{asf}(\mathbf{f}; \mathbf{w}) := \max \left\{ \frac{f'_i}{w_i} \right\}. \quad (12)$$

Although this scalarizing function is similar to TCH, ASF can find an objective vector parallel to  $w$ , improving diversity in MaOPs [18]. ASF has been employed in some MOEAs (e.g., [18, 50, 51]).

The **Augmented Achievement Scalarizing Function (AASF)** [13, p. 111] is a variation of ASF, where an extra term is considered for discarding weakly Pareto optimal solutions, and is defined as:

$$u^{aasf}(\mathbf{f}'; \mathbf{w}) := \max \left\{ \frac{f'_i}{w_i} \right\} + \alpha \sum_i \frac{f'_i}{w_i}. \quad (13)$$

$\alpha$  should take small values. In [8], it was recommended to set  $\alpha \approx 10^{-4}$ . There are few MOEAs that adopt this scalarizing function [8, 20, 52].

The **Penalty Boundary Intersection (PBI)** [17] draws ideas from the Normal-Boundary Intersection (NBI) method [53], defined as follows:

$$u^{pbi}(\mathbf{f}'; \mathbf{w}) := d_1 + \theta d_2 \quad (14)$$

where  $d_1 := \left| \mathbf{f}' \cdot \frac{\mathbf{w}}{\|\mathbf{w}\|} \right|$  and  $d_2 := \left\| \mathbf{f}' - d_1 \frac{\mathbf{w}}{\|\mathbf{w}\|} \right\|$ .

$\theta$  is a penalty parameter that balances convergence (measured by  $d_1$ ) and diversity (measured by  $d_2$ ), both should be minimized [54].  $d_2$  can be seen as the distance between  $\mathbf{f}'$  and its orthogonal projection on  $\mathbf{w}$ . When  $\theta = 0$ , the behavior of PBI is similar to WS. The PBI function can produce uniformly distributed solutions in objective space by setting an appropriate value for  $\theta$ . Several options for setting  $\theta$  have been studied considering the geometry of the Pareto front and the number of objectives [9, 54]. Recently, some attempts have been proposed for the adaptation of this parameter in MOEA/D [54].

The **Inverted Penalty Boundary Intersection (IPBI)** [55] is an extension of PBI, given by:

$$u^{ipbi}(\mathbf{f}'; \mathbf{w}) := \theta d_2 - d_1$$

where  $d_1 := \left| \mathbf{f}'' \cdot \frac{\mathbf{w}}{\|\mathbf{w}\|} \right|$  and  $d_2 := \left\| \mathbf{f}'' - d_1 \frac{\mathbf{w}}{\|\mathbf{w}\|} \right\|$ ,

where  $\mathbf{f}''$  is defined as:

$$\mathbf{f}''(\mathbf{x}) := \mathbf{z}^* - \mathbf{f}(\mathbf{x}), \quad (15)$$

and  $\mathbf{z}^* := (z_1^*, \dots, z_m^*)^T$  is the nadir point, i.e.,  $z_i^* := \max \{f_i(\mathbf{x}) \mid \mathbf{x} \in POS\}$ . The aim of IPBI is to enhance the spread of solutions in objective space and to improve the performance in MaOPs [55]. As in PBI,  $\theta$  handles the balance between  $d_1$  and  $d_2$ . However, a solution having a large  $d_1$  and a small  $d_2$  is considered as a better solution. In [11, 55], a set of different values for  $\theta$  were tested in MOEA/D for solving MaOPs in discrete and continuous spaces.

The **Conic Scalarization (CS)** [56] is a variant of WS, where an extra term is added for dealing with concave regions of the Pareto front. It is defined as:

$$u^{cs}(\mathbf{f}'; \mathbf{w}) := \sum_i w_i f'_i + \alpha \sum_i |f'_i|. \quad (16)$$

CS can generate weakly Pareto optimal solutions if  $\alpha \in [0, w_i]$ ,  $w_i > 0$  for all  $i \in \{1, \dots, m\}$  and there exists  $k \in \{1, \dots, m\}$  such that  $w_k > \alpha$ . There are few MOEAs that adopt CS (e.g., [57]).

The **Vector Angle Distance Scaling (VADS)** [28] can discover solutions in concavities that may appear as discontinuities in the Pareto front, given by:

$$u^{vads}(\mathbf{f}'; \mathbf{w}) := \frac{\|\mathbf{f}'\|}{\left( \frac{\mathbf{w}}{\|\mathbf{w}\|} \cdot \frac{\mathbf{f}'}{\|\mathbf{f}'\|} \right)^p}. \quad (17)$$

Here, the numerator measures convergence, whereas the denominator measures the deviation of the objective vector from the weight vector. Thus, the final solution should be lying parallel to  $\mathbf{w}$ . Orthogonal vectors require special care. Small

values of  $p$  hinder the search of sharp concavities. Authors in [28] recommend to use  $p = 100$ . This scalarizing function is not compatible with any form of Pareto optimality. VADS has been implemented on a MOEA in combination with TCH [28, 58].

### 3 Experimental Methodology

The goals of the experiments reported next are twofold: first, we want to determine which are the most suitable scalarizing functions for MOEA/D and MOMBI-II on different scenarios. These scenarios were built considering the Pareto front geometry and the number of objective functions. Second, we want to verify if this robust set of scalarizing functions can significantly improve the performance of the baseline versions of these MOEAs. To achieve the first target, we rely on a powerful tuning tool called Evolutionary Calibrator (EVOCA) [16], which can find good values for numerical and categorical parameters without requiring a strong knowledge of parameter tuning methods. The second goal is tackled by performing a comparative study using Nonparametric statistics. In the remainder of this section, we describe the test problems adopted for our experimental study, as well as the parameters settings of the MOEAs and further details of these experiments.

#### 3.1 Test Problems and Parameters Settings

We selected the Lamé Superspheres test problems [59] since they encompass the three basic Pareto front geometries in which the scalarizing functions present challenges. Moreover, this benchmark is scalable to any number of variables and objective functions. Hence, we tested for 2, 3, 5, 7 and 10 objectives ( $m$ ). The number of decision variables was set to  $n = m + 4$ . We fixed the parameter  $\gamma \in \{0.5, 1.0, 2.0\}$  to achieve Pareto fronts with convex, linear and concave geometries, respectively. Only unimodal problems were considered, since our aim was to determine if the scalarizing functions can handle different shapes of the Pareto front for multi- and many-objective problems. Thus, adding difficulties in the MOPs would introduce noise to the selection process of a MOEA.

The common parameters of MOEA/D and MOMBI-II adopted the same values. The population size was set to  $\{100, 120, 196, 210, 276\}$  individuals for  $\{2, 3, 5, 7, 10\}$  objectives, respectively. The probabilities of Polynomial-based mutation and Simulated Binary Crossover (SBX) were set to  $\frac{1}{n}$  and 0.9, respectively; in both cases, the distribution index was set to 20. The weight vectors were generated using the method described in [53], looking for a cardinality analogous to the population size. The stopping criterion consisted of reaching a maximum number of 50,000 evaluations of the MOP. The neighborhood size in MOEA/D was equal to 20. The parameter values employed for MOMBI-II were: record 5, tolerance threshold  $1 \times 10^{-3}$  and 0.5 for the variance threshold.

Our performance indicator was the *normalized hypervolume*, which is the hypervolume indicator [1] divided by the factor  $\prod_i r_i$ , where  $\mathbf{r} := (r_1, \dots, r_m)^T$  is a reference point, set to  $(2, \dots, 2)^T$  in all our test problems.

**Table 1.** EVOCA’s recommendation for each possible scenario. In every calibration, it is shown the scalarization function and its model parameter value (in parentheses).

$\gamma$	MOEA/D		MOMBI-II	
	Multi-objective	Many-objective	Multi-objective	Many-objective
<b>0.5</b>	AASF (0.8001)	EWC (7.6)	EWC (8.6) AASF (1.0)	TCH
<b>1.0</b>	PBI (15.1) PBI (20.5639)	AASF (0.3001)	PBI (21.4123)	AASF (0.3001)
<b>2.0</b>	PBI (2.6) PBI (7.6) VADS (8.6)	PBI (2.6)	PBI (2.4026)	VADS (5.6) VADS (2.1)
<b>Global</b>	ASF TCH	AASF (0.2423)	AASF (0.0501) TCH	ASF

### 3.2 Tuning Process

EVOCA [16] is a genetic algorithm whose aim is to find the parameter values of an optimizer that maximizes the profit for a given budget. In our case, the profit was the normalized hypervolume indicator, and the budget was limited to 10,000 executions of the optimizer.

An individual represents a calibration that involves one of the 15 scalarizing functions of Section 2, and the set of all the model parameters. To achieve accurate results, we consider that all the model parameters are real values. Based on our literature review, we selected the ranges:  $p \in [0.1, 10.0]$  for WCP, EWC, WPO, WN and VADS;  $\alpha \in [0.0001, 1.0]$  for ATCH, MTCH, AASF and CS;  $\theta \in [0.1, 50.0]$  for PBI and IPBI. Our main interest was to identify the scalarizing function(s) that solve a wide variety of test instances, or at least to know in which instances these scalarizing functions perform well. For this reason, we designed the following scenarios:

- Six scenarios that considered all the combinations of the cartesian product between the Pareto-front geometries, given by  $\gamma \in \{0.5, 1.0, 2.0\}$ , and the {multi, many} cases. The “multi” case groups 2 and 3 objectives, whereas the “many” considers 5, 7 and 10 objectives.
- A global scenario which includes all the previous combinations.

The tuning process was independent among scenarios and between MOEAs. At the end of a tuning process, EVOCA returned 20 individuals, corresponding to the best-found calibrations. We filtered these calibrations using Wilcoxon’s test (with a confidence level of 99%), executing 30 times each algorithm. The best-ranked calibrations are presented in Table 1.

In the multi-objective case, more than one calibration was obtained, but in the many-objective case, we found only one recommended scalarizing function. To summarize, EVOCA found that 6 out of 15 scalarizing functions (EWC, TCH, ASF, AASF, PBI, and VADS) had an outstanding performance in the particular scenarios that we studied. In the global scenario, EVOCA determined that TCH, ASF, and AASF had the best results, remarking that ASF forms part of the original version of MOMBI-II, while TCH is used by MOEA/D for 2 objectives.

**Table 2.** Median ( $\times 10^{-1}$ ) and standard deviation of the normalized hypervolume indicator on the **multi-objective scenarios**.

$\gamma$	MOEA/D			MOMBI-II		
	Config.	$m$		Config.	$m$	
		2	3		2	3
0.5	Baseline	9.570400 <sub>5.0e-07</sub>	9.906204 <sub>3.4e-04</sub>	Baseline	9.570404 <sub>6.0e-07</sub>	9.917509 <sub>1.2e-04</sub>
	AASF (0.8001)	<u>9.573699</u> <sub>4.5e-08</sub>	<u>9.974616</u> <sub>1.2e-05</sub>	EWC (8.6)	<u>9.575178</u> <sub>5.4e-08</sub>	<u>9.902034</u> <sub>6.8e-04</sub>
				AASF (1.0)	<u>9.573940</u> <sub>9.4e-08</sub>	<u>9.975483</u> <sub>1.3e-05</sub>
1.0	Baseline	8.737370 <sub>8.2e-08</sub>	9.744895 <sub>2.5e-07</sub>	Baseline	8.737369 <sub>8.0e-08</sub>	9.636011 <sub>6.3e-04</sub>
	PBI (15.1)	<u>8.737374</u> <sub>4.5e-08</sub>	9.744894 <sub>7.0e-07</sub>	PBI (21.4123)	<u>8.737374</u> <sub>4.5e-07</sub>	<u>9.744881</u> <sub>1.8e-06</sub>
	PBI (20.5639)	<u>8.737374</u> <sub>3.0e-07</sub>	9.744894 <sub>1.0e-06</sub>			
2.0	Baseline	8.025324 <sub>1.3e-07</sub>	9.277872 <sub>1.6e-06</sub>	Baseline	8.025323 <sub>9.6e-08</sub>	9.209490 <sub>1.0e-03</sub>
	PBI (2.6)	<u>8.025325</u> <sub>1.7e-07</sub>	<u>9.277875</u> <sub>1.1e-06</sub>	PBI (2.4026)	<u>8.025325</u> <sub>1.1e-07</sub>	<u>9.277874</u> <sub>1.0e-06</sub>
	PBI (7.6)	<u>8.025326</u> <sub>5.1e-07</sub>	9.277867 <sub>1.4e-06</sub>			
	VADS (8.6)	8.025324 <sub>1.4e-06</sub>	9.277872 <sub>1.5e-06</sub>			

Furthermore, the best options to solve problems with convex shape ( $\gamma = 0.5$ ) were AASF, EWC, and TCH. For linear shapes ( $\gamma = 1.0$ ) PBI and AASF. For convex shapes ( $\gamma = 2.0$ ) the best choices were PBI and VADS. Finally, the tuning process obtained a greater accuracy for the corresponding model parameters than that provided by the values recommended in the literature.

### 3.3 Comparative Study

In this subsection, we examine the performance of the calibrated versions given by EVOCA using the normalized hypervolume indicator (see Section 3.1). We performed 30 independent runs for all scenarios. We applied the Wilcoxon rank sum test (one-tailed) to the median of this indicator, to determine whether the calibrated version performed better than the baseline algorithm at a confidence interval of 99%. The baseline version adopted in MOEA/D was TCH for 2 objectives and PBI with  $\theta = 5$  for the remaining objectives [17, 60]. The baseline of MOMBI-II was ASF [18]. Both MOEAs and scalarizing functions were implemented in EMO Project,<sup>8</sup> a framework for Evolutionary Multi-Objective Optimization. This software is implemented in C language.

For all scenarios, our experimental results are shown in Tables 2, 3 and 4. The best value between the calibrated versions and the baseline MOEA is shown in gray. A line above the median ( $\bar{\cdot}$ ) implies that the calibrated version outperformed in a significant way the baseline algorithm. Conversely, a line under the median ( $\underline{\cdot}$ ) means that the calibrated version was significantly outperformed.

In the multi-objective scenarios of Table 2, we can observe a clear performance improvement over the calibrated versions for MOEA/D with 2 objectives and MOMBI-II with 2 and 3 objectives. Only the version of MOMBI-II using EWC (8.6) was outperformed by the baseline MOMBI-II. In the particular case of MOEA/D, the major gains were achieved for the convex MOPs, and the best suited scalarizing functions were AASF (0.8001) and PBI (15.1, 20.5639, 2.6,

<sup>8</sup> Available at <http://computacion.cs.cinvestav.mx/~rhernandez>

**Table 3.** Median ( $\times 10^{-1}$ ) and standard deviation of the normalized hypervolume indicator on the **many-objective scenarios**.

$\gamma$	Config.	$m$		
		5	7	10
<b>MOEA/D</b>				
<b>0.5</b>	Baseline	9.961741 8.0e-04	9.908978 1.4e-03	9.831210 1.1e-03
	EWC (7.6)	<u>9.999756</u> 6.1e-08	<u>9.999988</u> 1.8e-08	<u>9.999985</u> 1.1e-16
<b>1.0</b>	Baseline	9.989021 2.1e-07	9.999387 1.9e-06	9.999864 1.5e-05
	AASF (0.3001)	<u>9.989041</u> 5.7e-07	<u>9.999427</u> 3.8e-07	<u>9.999990</u> 5.9e-07
<b>2.0</b>	Baseline	9.904560 2.3e-06	9.986141 4.5e-06	9.999186 4.2e-06
	PBI (2.6)	<u>9.904578</u> 1.8e-06	<u>9.986145</u> 5.4e-06	9.999160 3.8e-06
<b>MOMBI-II</b>				
<b>0.5</b>	Baseline	9.970399 3.9e-04	9.987786 3.3e-04	9.991083 2.8e-04
	TCH	<u>9.999357</u> 3.2e-06	<u>9.999964</u> 1.5e-06	<u>9.999942</u> 4.1e-06
<b>1.0</b>	Baseline	9.937457 9.2e-04	9.977353 8.9e-04	9.993752 4.5e-04
	AASF (0.3001)	<u>9.989052</u> 5.6e-07	<u>9.999427</u> 4.8e-08	<u>9.999991</u> 0.0e+0
<b>2.0</b>	Baseline	9.884399 3.9e-04	9.976128 2.2e-04	9.995263 1.5e-04
	VADS (5.6)	<u>9.904381</u> 6.7e-06	<u>9.985715</u> 1.9e-05	<u>9.998632</u> 1.7e-05
	VADS (2.1)	9.904350 8.6e-06	9.985683 2.4e-05	9.998591 2.0e-05

7.6). In MOMBI-II, the major gains were in the convex MOPs and the remaining problems with 3 objectives. The best scalarizing functions for this optimizer were EWC (8.6), AASF (1.0) and PBI (21.4123, 2.4026). As it can be noticed, AASF worked very well for both MOEAs in the convex problems, while PBI performed best in the linear and concave problems. However, this scalarizing function is sensitive to its parameter value.

In the many-objective scenarios of Table 3, we can notice a clear performance improvement over the calibrated versions of both optimizers. In MOEA/D, there were only 2 ties for the concave problems, and the major gains were in the convex MOPs. The best scalarizing functions for MOEA/D were: EWC (7.6), AASF (0.3001) and PBI (2.6). In MOMBI-II, the major gains were in 5 and 7 objectives. The best scalarizing functions for this optimizer were TCH, AASF (0.3001), VADS (5.6, 2.1). In both optimizers, AASF (0.3001) worked very well on the linear problems.

In the global scenario of Table 4 a different pattern is observed. In the case of MOEA/D, ASF and TCH worked very well in the convex problems from 3 up to 10 objectives. However, they worsened their behavior in the linear and concave MOPs. Similarly, AASF (0.2423) performed well in the convex problems and the linear problems for 5, 7 and 10 objectives. However, its performance deteriorated in the concave MOPs. On the other hand, for MOMBI-II, the 2 scalarizing functions were complementary to each other. For example, for 2 objectives, TCH was competitive with respect to the baseline version, while in the concave problems for 3 to 10 objectives AASF (0.0501) performed best.

These results suggest that no scalarizing function can solve effectively all the problems. Instead, there is a subset of them that can tackle in an effective manner some specific problems.

**Table 4.** Median ( $\times 10^{-1}$ ) and standard deviation of the normalized hypervolume indicator on the **global scenario**.

$\gamma$	Config.	$m$									
		2	3	5	7	10					
<b>MOEA/D</b>											
0.5	Baseline	9.570400	5.0e-07	9.906204	3.4e-04	9.961741	8.0e-04	9.908978	1.4e-03	9.831210	1.1e-03
	ASF	9.570399	3.2e-07	9.917019	1.5e-04	9.971177	5.7e-04	9.987843	2.6e-04	9.994325	2.8e-04
	TCH	9.570400	5.0e-07	9.953880	2.6e-06	9.999339	1.6e-06	9.999972	5.5e-06	9.999967	6.3e-06
	AASF (0.2423)	9.571983	1.9e-07	9.966615	7.4e-05	9.995188	5.0e-05	9.997490	4.8e-05	9.998646	5.4e-05
1.0	Baseline	8.737370	8.2e-08	9.744895	2.5e-07	9.989021	2.1e-07	9.999387	1.9e-06	9.999864	1.5e-05
	ASF	8.737369	7.2e-08	9.638966	6.2e-04	9.914455	7.2e-04	9.963243	8.9e-04	9.988689	4.1e-04
	TCH	8.737370	8.2e-08	9.689283	1.8e-05	9.967928	2.4e-05	9.983080	4.6e-04	9.932911	2.2e-03
	AASF (0.2423)	8.668011	1.3e-06	9.720175	4.1e-05	9.989038	2.8e-07	9.999426	2.8e-07	9.999991	4.9e-07
2.0	Baseline	8.025324	1.3e-07	9.277872	1.6e-06	9.904560	2.3e-06	9.986141	4.5e-06	9.999186	4.2e-06
	ASF	8.025323	1.0e-07	9.207519	7.9e-04	9.870073	6.6e-04	9.970228	5.6e-04	9.985247	1.6e-03
	TCH	8.025324	1.3e-07	9.228448	5.7e-05	9.854362	1.3e-05	9.807065	2.3e-03	9.561658	6.5e-03
	AASF (0.2423)	7.876793	2.0e-06	9.152424	2.5e-03	9.847468	2.3e-03	9.969067	5.6e-04	9.996733	1.3e-04
<b>MOMBI-II</b>											
0.5	Baseline	9.570404	6.0e-07	9.917509	1.2e-04	9.970399	3.9e-04	9.987786	3.3e-04	9.991083	2.8e-04
	TCH	9.570404	6.2e-07	9.953905	3.6e-06	9.999357	3.2e-06	9.999964	1.5e-06	9.999942	4.1e-06
	AASF (0.0501)	9.570326	4.4e-07	9.924333	2.7e-04	9.979430	3.4e-04	9.989421	2.1e-04	9.993698	2.2e-04
1.0	Baseline	8.737369	8.0e-08	9.636011	6.3e-04	9.937457	9.2e-04	9.977353	8.9e-04	9.993752	4.5e-04
	TCH	8.737369	7.2e-08	9.689161	1.2e-05	9.968534	1.1e-04	9.984665	6.2e-04	9.931828	2.4e-03
	AASF (0.0501)	8.732288	1.2e-05	9.744887	6.7e-07	9.989033	2.2e-07	9.999427	1.8e-08	9.999991	1.0e+0
2.0	Baseline	8.025323	9.6e-08	9.209490	1.0e-03	9.884399	3.9e-04	9.976128	2.2e-04	9.995263	1.5e-04
	TCH	8.025323	9.6e-08	9.229191	1.5e-04	9.855497	7.7e-04	9.851832	2.2e-03	9.563050	6.3e-03
	AASF (0.0501)	8.003164	2.5e-05	9.269135	1.0e-04	9.898159	1.1e-04	9.985653	1.7e-05	9.999221	5.8e-06

## 4 Conclusions and Future Work

In this work, we presented an overview and a comparative study to determine the weighted and unconstrained scalarizing functions that are the most suitable for MOEA/D and MOMBI-II. For this purpose, we designed several test scenarios considering different Pareto-front shapes and objectives. We used the tuning tool EVOCA to determine the best calibration for each of these scenarios. In almost all cases, EVOCA recommendations outperform the baseline version of these MOEAs. Our most important conclusion is that no single scalarizing function performs best in all the scenarios but a set of them, regarding the normalized hypervolume indicator. In general, we obtained good results with AASF, PBI, EWC, and VADS. These scalarizing functions deserve further study.

As part of our future work, we are interested in studying additional scenarios that consider more difficult problems including other performance indicators. We are interested in analyzing the effect of using the nadir point instead of the ideal point (as in IPBI) and using vectors formed by the inverse components of the weight vectors (as in ASF). From the obtained results, we would like to design self-adaptive models and new MOEAs that combine several scalarizing functions.

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