A

MOPs in the Literature

This appendix contains the tables classifying many of the known MOEA test functions as of early 2007.[1] Tables A.1, A.2, A.3 and A.4 contain unconstrained MOPs, and Section A.5 contains the description of a (new) benchmark of scalable test functions proposed by Huband et al. [693, 694]. Table A.5 contains constrained MOPs.[2]

As previously discussed, together, all of these test functions form a *de facto* MOEA test function suite.

Appendix B contains graphical representations of P_{true} and PF_{true} for the MOPs of Tables A.1. Appendix C contains graphical representations of P_{true} and PF_{true} for the MOPs of Tables A.2. Appendix D contains graphical representations of PF_{true} for the MOPs of Tables A.3. Appendix E contains graphical representations of PF_{true} for the MOPs of Tables A.4. Appendix F contains graphical representations of PF_{true} for the (unconstrained) MOPs described in Section A.5. Appendix G contains graphical representations of P_{true} and PF_{true} for the MOPs of Table A.5.

Please note that the data files containing PF_{true} for most of the problems described in these Appendices (except for those test functions for which PF_{true} has an analytical form), are available for download at:

`http://www.cs.cinvestav.mx/~emoobook`

[1] Because several distinct MOPs may be created using Deb's initial methodology [341], direct implementations of those functions are not listed here.

[2] Special thanks to Antonio Nebro and Enrique Alba (from the University of Málaga, in Spain) for providing the true Pareto fronts of many of the test functions included in this Appendix.

A.1 MOP Numeric Unconstrained Test Functions (Part I)

Table A.1: MOP Numeric Unconstrained Test Functions (Part I)

Researcher & Major MOP Characteristics	Definition	Constraints
Binh (1) [121, 123]; P_{true} connected, PF_{true} convex	$F = (f_1(x,y), f_2(x,y))$, where $$f_1(x,y) = x^2 + y^2, \quad f_2(x,y) = (x-5)^2 + (y-5)^2$$	$-5 \leq x, y \leq 10$
Binh (3) [120];	$F = (f_1(x,y), f_2(x,y), f_3(x,y))$, where $$f_1(x,y) = x - 10^6, \quad f_2(x,y) = y - 2 * 10^{-6}, \quad f_3(x,y) = xy - 2$$	$10^{-6} \leq x, y \leq 10^6$
Fonseca [485]; P_{true} connected, PF_{true} concave	$F = (f_1(x,y), f_2(x,y))$, where $$f_1(x,y) = 1 - \exp(-(x-1)^2 - (y+1)^2), \quad f_2(x,y) = 1 - \exp(-(x+1)^2 - (y-1)^2)$$	None
Fonseca (2) [483]; P_{true} connected, PF_{true} concave, Analytical solution stated	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where $$f_1(\mathbf{x}) = 1 - \exp(-\sum_{i=1}^{n}(x_i - \frac{1}{\sqrt{n}})^2), \quad f_2(\mathbf{x}) = 1 - \exp(-\sum_{i=1}^{n}(x_i + \frac{1}{\sqrt{n}})^2)$$	$-4 \leq x_i \leq 4$
Kursawe (1) [900]; P_{true} disconnected, PF_{true} disconnected	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where $$f_1(\mathbf{x}) = \sum_{i=1}^{n-1}(-10e^{(-0.2)*\sqrt{x_i^2 + x_{i+1}^2}}), \quad f_2(\mathbf{x}) = \sum_{i=1}^{n}(\lvert x_i\rvert^{0.8} + 5sin(x_i)^3)$$	None
Laumanns [923]; P_{true} disconnected, PF_{true} convex	$F = (f_1(x,y), f_2(x,y))$, where $$f_1(x,y) = x^2 + y^2, \quad f_2(x,y) = (x+2)^2 + y^2$$	$-50 \leq x, y \leq 50$

Table A.1: (continued)

Researcher & Major MOP Characteristics	**Definition**	**Constraints**
Lis [963]; P_{true} disconnected, PF_{true} disconnected and concave	$F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = \sqrt[8]{x^2+y^2}$, $f_2(x,y) = \sqrt[4]{(x-0.5)^2+(y-0.5)^2}$	$-5 \leq x,y \leq 10$
Murata [1497, 1106]; P_{true} connected, PF_{true} concave	$F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = 2\sqrt{x}$, $f_2(x,y) = x(1-y)+5$	$1 \leq x \leq 4,\ 1 \leq y \leq 2$
Poloni [1236, 1235]; P_{true} disconnected, PF_{true} disconnected and convex	Maximize $F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = -[1+(A_1-B_1)^2+(A_2-B_2)^2]$, $f_2(x,y) = -[(x+3)^2+(y+1)^2]$	$-\pi \leq x,y \leq \pi$, $A_1 = 0.5\sin 1 - 2\cos 1 + \sin 2 - 1.5\cos 2$, $A_2 = 1.5\sin 1 - \cos 1 + 2\sin 2 - 0.5\cos 2$, $B_1 = 0.5\sin x - 2\cos x + \sin y - 1.5\cos y$, $B_2 = 1.5\sin x - \cos x + 2\sin y - 0.5\cos y$,
Quagliarella [1256]; P_{true} disconnected, PF_{true} convex	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where $f_1(\mathbf{x}) = \sqrt{\frac{A_1}{n}}$, $f_2(\mathbf{x}) = \sqrt{\frac{A_2}{n}}$	$A_1 = \sum_{i=1}^{n}[(x_i)^2 - 10\cos[2\pi(x_i)] + 10]$, $A_2 = \sum_{i=1}^{n}[(x_i-1.5)^2 - 10\cos[2\pi(x_i - 1.5)] + 10]$, $-5.12 \leq x_i \leq 5.12, n = 16$
Rendon [1565]; P_{true} connected, PF_{true} convex	$F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = \frac{1}{x^2+y^2+1}$, $f_2(x,y) = x^2+3y^2+1$	$-3 \leq x,y \leq 3$

Table A.1: (continued)

Researcher & Major MOP Characteristics	Definition	Constraints
Rendon (2) [1565]; P_{true} connected, PF_{true} convex	$F = (f_1(x,y), f_2(x,y))$, where $$f_1(x,y) = x + y + 1,$$ $$f_2(x,y) = x^2 + 2y - 1$$	$-3 \leq x, y \leq 3$
Schaffer [777, 1144, 1383]; P_{true} connected, PF_{true} convex, Analytical solution proved [1341]	$F = (f_1(x), f_2(x))$, where $$f_1(x) = x^2,$$ $$f_2(x) = (x-2)^2$$	None
Schaffer (2) [1451, 108]; P_{true} disconnected, PF_{true} disconnected	$F = (f_1(x), f_2(x))$, where $$f_1(x) = -x, \text{ if } x \leq 1,$$ $$= -2 + x, \text{ if } 1 < x \leq 3,$$ $$= 4 - x, \text{ if } 3 < x \leq 4,$$ $$= -4 + x, \text{ if } x > 4,$$ $$f_2(x) = (x-5)^2$$	$-5 \leq x \leq 10$
Viennet [1592]; P_{true} connected and symmetric, PF_{true} curved surface	$F = (f_1(x,y), f_2(x,y), f_3(x,y))$, where $$f_1(x,y) = x^2 + (y-1)^2,$$ $$f_2(x,y) = x^2 + (y+1)^2 + 1,$$ $$f_3(x,y) = (x-1)^2 + y^2 + 2$$	$-2 \leq x, y \leq 2$
Viennet (2) [1592]; P_{true} connected, PF_{true} disconnected	$F = (f_1(x,y), f_2(x,y), f_3(x,y))$, where $$f_1(x,y) = \frac{(x-2)^2}{2} + \frac{(y+1)^2}{13} + 3,$$ $$f_2(x,y) = \frac{(x+y-3)^2}{36} + \frac{(-x+y+2)^2}{8}$$ $$-17,$$ $$f_3(x,y) = \frac{(x+2y-1)^2}{175} + \frac{(2y-x)^2}{17} - 13$$	$-4 \leq x, y \leq 4$
Viennet (3) [1592]; P_{true} disconnected and unsymmetric, PF_{true} connected	$F = (f_1(x,y), f_2(x,y), f_3(x,y))$, where $$f_1(x,y) = 0.5 * (x^2 + y^2) + sin(x^2 + y^2),$$ $$f_2(x,y) = \frac{(3x-2y+4)^2}{8} + \frac{(x-y+1)^2}{27}$$ $$+15,$$ $$f_3(x,y) = \frac{1}{(x^2+y^2+1)} - 1.1e^{(-x^2-y^2)}$$	$-3 \leq x, y \leq 3$

A.2 MOP Numeric Unconstrained Test Functions (Part II)

Table A.2: MOP Numeric Unconstrained Test Functions (Part II)

Researcher & Major MOP Characteristics	Definition	Constraints
Deb [341]; P_{true} connected, PF_{true} connected	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = x_1, \quad f_2(\mathbf{x}, g) = g(\mathbf{x}) \cdot h(\mathbf{x}),$$ and: $$g(\mathbf{x}) = 1 + x_2^2, \quad h(\mathbf{x}) = \begin{cases} 1 - \left(\frac{f_1(\mathbf{x})}{g(\mathbf{x})}\right)^2, & \text{if } f_1 \leq g; \\ 0, & \text{otherwise} \end{cases}$$	$0 \leq x_i \leq 1$, $i = 1, 2$
Deb (2) [341]; P_{true} disconnected; PF_{true} disconnected	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = x_1, \quad f_2(\mathbf{x}, g) = g(\mathbf{x}) \cdot h(\mathbf{x}),$$ and: $$g(\mathbf{x}) = 1 + 10 \cdot x_2, \quad h(\mathbf{x}) = 1 - \left(\frac{f_1}{g(\mathbf{x})}\right)^2 - \frac{f_1}{g(\mathbf{x})} \cdot \sin(12\pi f_1)$$	$0 \leq x_i \leq 1$, $i = 1, 2$
Deb (3) [341]; P_{true} connected; PF_{true} connected	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = 1 - e^{-4x_1} \sin^4(10\pi x_1), \quad f_2(\mathbf{x}, g) = g(\mathbf{x}) \cdot h(\mathbf{x}),$$ and: $$g(\mathbf{x}) = 1 + x_2^2, \quad h(\mathbf{x}) = \begin{cases} 1 - \left(\frac{f_1(\mathbf{x})}{g(\mathbf{x})}\right)^{10}, & \text{if } f_1 \leq g; \\ 0, & \text{otherwise} \end{cases}$$	$0 \leq x_i \leq 1$, $i = 1, 2$
OKA 1 [1161];	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = x_1', \quad f_2(\mathbf{x}) = \sqrt{2\pi} - \sqrt{\lvert x_1' \rvert} + 2\lvert x_2' - 3\cos(x_1') - 3\rvert^{\frac{1}{2}}$$ $$x_1' = \cos(\pi/12)x_1 - \sin(\pi/12)x_2, \quad x_2' = \sin(\pi/12)x_1 + \cos(\pi/12)x_2$$	$s_1 = \sin(\pi/12)$, $s_2 = \cos(\pi/12)$, $6s_1 \leq x_1 \leq 6s_1 + 2\pi s_2$, $-2\pi s_1 \leq x_2 \leq 6s_2$

Table A.2: (continued)

Researcher & Major MOP Characteristics	**Definition**	**Constraints**
OKA 2 [1161];	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where: $$\begin{aligned} f_1(\mathbf{x}) &= x_1 \\ f_2(\mathbf{x}) &= 1 - \frac{1}{4\pi^2}(x_1+\pi)^2 \\ &\quad + \lvert x_2 - 5\cos(x_1)\rvert^{\frac{1}{3}} \\ &\quad + \lvert \tfrac{x}{3} - 5\sin(x_1)\rvert^{\frac{1}{3}} \end{aligned}$$	$-\pi \leq x_1 \leq \pi$ $-5 \leq x_2 \leq 5$

A.3 MOP Numeric Unconstrained Test Functions (Part III)

Table A.3: MOP Numeric Unconstrained Test Functions (Part III)

Researcher & Major MOP Characteristics	Definition	Constraints
ZDT1 [1704]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = x_1,$$ $$f_2(\mathbf{x}, g) = g(\mathbf{x}) \cdot (1 - \sqrt{\frac{f_1}{g(\mathbf{x})}})$$ and: $$g(\mathbf{x}) = 1 + \frac{9}{n-1} \cdot \sum_{i=2}^{n} x_i$$	$n = 30,$ $0 \leq x_i \leq 1,$ $i = 1, 2, \ldots, 30$
ZDT2 [1704]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = x_1,$$ $$f_2(\mathbf{x}, g) = g(\mathbf{x}) \cdot (1 - (\frac{f_1}{g(\mathbf{x})})^2)$$ and: $$g(\mathbf{x}) = 1 + \frac{9}{n-1} \cdot \sum_{i=2}^{n} x_i$$	$n = 30,$ $0 \leq x_i \leq 1,$ $i = 1, 2, \ldots, 30$
ZDT3 [1704]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = x_1$$ $$f_2(\mathbf{x}, g) = g(\mathbf{x}) \cdot (1 - \sqrt{\frac{f_1}{g(\mathbf{x})}} - \frac{f_1}{g(\mathbf{x})} \cdot \sin(10\pi f_1))$$ and: $$g(\mathbf{x}) = 1 + \frac{9}{n-1} \cdot \sum_{i=2}^{n} x_i$$	$n = 30,$ $0 \leq x_i \leq 1,$ $i = 1, 2, \ldots, 30$
ZDT4 [1704]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = x_1$$ $$f_2(\mathbf{x}, g) = g(\mathbf{x}) \cdot (1 - \sqrt{\frac{f_1}{g(\mathbf{x})}})$$ and: $$g(\mathbf{x}) = 1 + 10 \cdot (n-1) + \sum_{i=2}^{n}(x_i^2 - 10\cos(4\pi x_i))$$	$n = 10,$ $0 \leq x_1 \leq 1,$ $-5 \leq x_i \leq 5,$ $i = 2, 3, \ldots, 10$

Table A.3: (continued)

Researcher & Major MOP Characteristics	**Definition**	**Constraints**
ZDT5 [1704]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = 1 + u(x_1)$$ $$f_2(\mathbf{x}, g) = \frac{g(\mathbf{x})}{f_1(\mathbf{x})}$$ and: $$g(\mathbf{x}) = \sum_{i=2}^{11} v(u(x_i)),$$ $$v(u(x_i)) = \begin{cases} 2 + u(x_i), & \text{if } u(x_i) < 5; \\ 1, & \text{if } u(x_i) = 5. \end{cases}$$	x_1 is represented by a 30-bit substring and the 10 remaining variables ($x_2 - x_{11}$) are represented by a 5 bits substring each. The function $u(x_i)$ denotes the number of 1's in the substring used to represent the variable x_i;
ZDT6 [1704]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = 1 - \exp(-4x_1) \cdot \sin^6(6\pi x_1)$$ $$f_2(\mathbf{x}, g) = g(\mathbf{x}) \cdot (1 - (\frac{f_1}{g(\mathbf{x})})^2)$$ and: $$g(\mathbf{x}) = 1 + 9 \cdot [\frac{(\sum_{i=2}^{n} x_i)}{9}]^{0.25}$$	$$n = 10,$$ $$0 \leq x_i \leq 1,$$ $$i = 1, 2, \ldots, 10$$

A.4 MOP Numeric Unconstrained Test Functions (Part IV)

Table A.4: MOP Numeric Unconstrained Test Functions (Part IV)

Researcher & Major MOP Characteristics	Definition	Constraints
DTLZ1 [361, 362]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = \frac{1}{2}x_1x_2(1+g(\mathbf{x}))$$ $$f_2(\mathbf{x}) = \frac{1}{2}x_1(1-x_2)(1+g(\mathbf{x}))$$ $$f_2(\mathbf{x}) = \frac{1}{2}(1-x_1)(1+g(\mathbf{x}))$$ and: $$g(\mathbf{x}) = 100[10 + \sum_{i=3}^{n}(x_i-0.5)^2 - \cos(20\pi(x_i-0.5))]$$	$n = 12,$ $0 \leq x_i \leq 1,$ $i = 1, ..., 12$
DTLZ2 [361, 362]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = \cos(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2)(1+g(\mathbf{x}))$$ $$f_2(\mathbf{x}) = \cos(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2)(1+g(\mathbf{x}))$$ $$f_3(\mathbf{x}) = \sin(\frac{\pi}{2}x_1)(1+g(\mathbf{x}))$$ and: $$g(\mathbf{x}) = \sum_{i=3}^{n}(x_i-0.5)^2$$	$n = 12,$ $0 \leq x_i \leq 1,$ $i = 1, ..., 12$
DTLZ3 [361, 362]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = \cos(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2)(1+g(\mathbf{x}))$$ $$f_2(\mathbf{x}) = \cos(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2)(1+g(\mathbf{x}))$$ $$f_3(\mathbf{x}) = \sin(\frac{\pi}{2}x_1)(1+g(\mathbf{x}))$$ and: $$g(\mathbf{x}) = 100[10 + \sum_{i=3}^{12}(x_i-0.5)^2 - \cos(20\pi(x_i-0.5))]$$	$n = 12,$ $0 \leq x_i \leq 1,$ $i = 1, ..., 12$

Table A.4: (continued)

Researcher & Major MOP Characteristics	Definition	Constraints
DTLZ4 [361, 362]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = \cos(\frac{\pi}{2}x_1^{\alpha})\cos(\frac{\pi}{2}x_2^{\alpha})(1+g(\mathbf{x}))$$ $$f_2(\mathbf{x}) = \cos(\frac{\pi}{2}x_1^{\alpha})\sin(\frac{\pi}{2}x_2^{\alpha})(1+g(\mathbf{x}))$$ $$f_3(\mathbf{x}) = \sin(\frac{\pi}{2}x_1^{\alpha})(1+g(\mathbf{x}))$$ and: $$g(\mathbf{x}) = \sum_{i=3}^{n}(x_i-0.5)^2$$	$n = 12$, $\alpha = 100$, $0 \leq x_i \leq 1$, $i = 1, \ldots, 12$
DTLZ5 [361, 362]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = \cos(\frac{\pi}{2}\theta_1)\cos(\frac{\pi}{2}\theta_2)(1+g(\mathbf{x}))$$ $$f_2(\mathbf{x}) = \cos(\frac{\pi}{2}\theta_1)\sin(\frac{\pi}{2}\theta_2)(1+g(\mathbf{x}))$$ $$f_3(\mathbf{x}) = \sin(\frac{\pi}{2}\theta_1)(1+g(\mathbf{x}))$$ and: $$\theta_1 = x_1 \cdot \left(\frac{\pi}{2}\right),$$ $$\theta_2 = \frac{\pi}{4\cdot(1+g(\mathbf{x}))}\cdot(1+2x_2\cdot g(\mathbf{x})),$$ $$g(\mathbf{x}) = \sum_{i=3}^{n}(x_i-0.5)^2$$	$n = 12$, $0 \leq x_i \leq 1$, $i = 1, \ldots, 12$
DTLZ6 [361, 362]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$, where: $$f_1(\mathbf{x}) = \cos(\frac{\pi}{2}\theta_1)\cos(\frac{\pi}{2}\theta_2)(1+g(\mathbf{x}))$$ $$f_2(\mathbf{x}) = \cos(\frac{\pi}{2}\theta_1)\sin(\frac{\pi}{2}\theta_2)(1+g(\mathbf{x}))$$ $$f_3(\mathbf{x}) = \sin(\frac{\pi}{2}\theta_1)(1+g(\mathbf{x}))$$ and: $$\theta_2 = \frac{\pi}{4\cdot(1+g(\mathbf{x}))}\cdot(1+2x_i\cdot g(\mathbf{x})),$$ $$g(\mathbf{x}) = \sum_{i=3}^{n}(x_i)^{0.1}$$	$n = 12$, $0 \leq x_i \leq 1$, $i = 1, \ldots, 12$

Table A.4: (continued)

Researcher & Major MOP Characteristics	**Definition**	**Constraints**
DTLZ7 [361, 362]	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$, where: $f_1(\mathbf{x}) = x_1$ $f_2(\mathbf{x}) = x_2$ $f_3(\mathbf{x}) = (1 + g(\mathbf{x})) \cdot h(f_1, f_2, g(\mathbf{x}))$ and: $g(\mathbf{x}) = 1 + \frac{9}{22} \cdot \sum_{i=3}^{n} (x_i)$ $h(f_1, f_2, g) = 3 - \sum_{i=1}^{2} \left(\frac{f_i}{1+g}(1 + \sin(3\pi f_i)) \right)$	$n = 22,$ $0 \leq x_i \leq 1,$ $i = 1, \ldots, 12$

A.5 MOP Numeric Unconstrained Test Functions (Part V)

Huband et al. [693, 694] recently proposed a new benchmark to validate multi-objective evolutionary algorithms. The description of these test functions is provided next.

A.5.1 Shapes for the objective functions

$$\begin{aligned} \text{linear}_1(x_1,\ldots,x_{M-1}) &= \prod_{i=1}^{M-1} x_i \\ \text{linear}_{m=2:M-1}(x_1,\ldots,x_{M-1}) &= \left(\prod_{i=1}^{M-m} x_i\right)(1-x_{M-m+1}) \\ \text{linear}_M(x_1,\ldots,x_{M-1}) &= 1-x_1 \end{aligned}$$

$$\begin{aligned} \text{convex}_1(x_1,\ldots,x_{M-1}) &= \prod_{i=1}^{M-1}(1-\cos(x_i\pi/2)) \\ \text{convex}_{m=2:M-1}(x_1,\ldots,x_{M-1}) &= \left(\prod_{i=1}^{M-m}(1-\cos(x_i\pi/2))\right)(1-\sin(x_{M-m+1}\pi/2)) \\ \text{convex}_M(x_1,\ldots,x_{M-1}) &= 1-\sin(x_1\pi/2) \end{aligned}$$

$$\begin{aligned} \text{concave}_1(x_1,\ldots,x_{M-1}) &= \prod_{i=1}^{M-1}\sin(x_i\pi/2)) \\ \text{concave}_{m=2:M-1}(x_1,\ldots,x_{M-1}) &= \left(\prod_{i=1}^{M-m}\sin(x_i\pi/2)\right)\cos(x_{M-m+1}\pi/2) \\ \text{concave}_M(x_1,\ldots,x_{M-1}) &= \cos(x_1\pi/2) \end{aligned}$$

$$\text{mixed}_M(x_1,\ldots,x_{M-1}) = \left(1-x_1-\frac{\cos(2A\pi x_1+\pi/2)}{2A\pi}\right)^{\alpha}$$

$$\text{disc}_M(x_1,\ldots,x_{M-1}) = 1-x_1^{\alpha}\cos^2(Ax_1^{\beta}\pi)$$

Transformation functions

$$\text{b\_poly}(y,\alpha) = y^{\alpha}$$

$$\text{b\_flat}(y,A,B,C) = A+\min(0,\lfloor y-B\rfloor)\frac{A(B-y)}{B}-\min(0,\lfloor C-y\rfloor)\frac{(1-A)(y-C)}{1-C}$$

$$\text{b\_param}(y, u(\mathbf{y}'), A, B, C) = y^{B+(C-B)\left(A-(1-2u(\mathbf{y}'))\left|\lfloor 0.5-u(\mathbf{y}')\rfloor + A\right|\right)}$$

$$\text{s\_linear}(y, A) = \frac{|y - A|}{|\lfloor A - y\rfloor + A|}$$

$$\text{s\_decept}(y, A, B, C) = 1 + (|y - A| - B)\left(\frac{\lfloor y - A + B\rfloor\left(1 - C + \frac{A-B}{B}\right)}{A - B} + \frac{\lfloor A + B - y\rfloor\left(1 - C + \frac{1-A-B}{B}\right)}{1 - A - B} + \frac{1}{B}\right)$$

$$\text{s\_multi}(y, A, B, C) = \frac{1 + \cos\left((4A+2)\pi\left(0.5 - \frac{|y-C|}{2(\lfloor C-y\rfloor + C)}\right)\right) + 4B\left(\frac{|y-C|}{2(\lfloor C-y\rfloor + C)}\right)^2}{b+2}$$

$$\text{r\_sum}(\mathbf{y}, \mathbf{w}) = \frac{\sum_{i=1}^{|\mathbf{y}|} w_1 y_i}{\sum_{i=1}^{|\mathbf{y}|} w_i}$$

$$\text{r\_nonsep}(\mathbf{y}, A) = \frac{\sum_{j=1}^{|\mathbf{y}|}\left(y_j + \sum_{k=0}^{A-2}\left|y_j - y_{1+(j+k) mod |\mathbf{y}|}\right|\right)}{\frac{|\mathbf{y}|}{A}\left\lceil\frac{A}{2}\right\rceil\left(1 + 2A - 2\left\lceil\frac{A}{2}\right\rceil\right)}$$

A.5.2 Test Problems

WFG1

Minimize

$$\begin{aligned} f_{m=1:M-1}(\mathbf{x}) &= x_M + S_m \text{convex}_m(x_1, \ldots, x_{M-1}) \\ f_M(\mathbf{x}) &= x_M + S_M \text{mixed}_M(x_1, \ldots, x_{M-1}) \end{aligned}$$

where

$$\begin{aligned} y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [2((i-1)k/(M-1)+1), \ldots, 2ik/(M-1)]) \\ y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_n], [2(k+1), \ldots, 2n]) \\ y'_{i=1:n} &= \text{b\_poly}(y''_i, 0.02) \\ y''_{i=1:k} &= y'''_i \\ y''_{i=k+1:n} &= \text{b\_flat}(y'''_i, 0.8, 0.75, 0.85) \\ y'''_{i=1:k} &= z_{i,[0,1]} \\ y'''_{i=k+1:n} &= \text{s\_linear}(z_{i,[0,1]}, 0.35) \end{aligned}$$

WFG2

Minimize

$$\begin{aligned} f_{m=1:M-1}(\mathbf{x}) &= x_M + S_m \text{convex}_m(x_1, \ldots, x_{M-1}) \\ f_M(\mathbf{x}) &= x_M + S_M \text{disc}_M(x_1, \ldots, x_{M-1}) \end{aligned}$$

where

$$\begin{aligned} y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\ y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_{k+l/2}], [1, \ldots, 1]) \\ y'_{i=1:k} &= y''_i \\ y'_{i=k+1:k+l/2} &= \text{r\_nonsep}([y''_{k+2(i-k)-1}, y''_{k+2(i-k)}], 2) \\ y''_{i=1:k} &= z_{i,[0,1]} \\ y''_{i=k+1:n} &= \text{s\_linear}(z_{i,[0,1]}, 0.35) \end{aligned}$$

WFG3

Minimize

$$f_{m=1:M}(\mathbf{x}) = x_M + S_m \text{linear}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned} y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\ y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_{k+l/2}], [1, \ldots, 1]) \\ y'_{i=1:k} &= y''_i \\ y'_{i=k+1:k+l/2} &= \text{r\_nonsep}([y''_{k+2(i-k)-1}, y''_{k+2(i-k)}], 2) \\ y''_{i=1:k} &= z_{i,[0,1]} \\ y''_{i=k+1:n} &= \text{s\_linear}(z_{i,[0,1]}, 0.35) \end{aligned}$$

WFG4

Minimize

$$f_{m=1:M}(\mathbf{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$
\begin{aligned}
y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\
y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\
y'_{i=1:n} &= \text{s\_multi}(z_{i,[0,1]}, 30, 10, 0.35)
\end{aligned}
$$

WFG5

Minimize

$$f_{m=1:M}(\mathbf{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$
\begin{aligned}
y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\
y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\
y'_{i=1:n} &= \text{s\_decept}(z_{i,[0,1]}, 0.35, 0.001, 0.05)
\end{aligned}
$$

WFG6

Minimize

$$f_{m=1:M}(\mathbf{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$
\begin{aligned}
y_{i=1:M-1} &= \text{r\_nonsep}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], k/(M-1)) \\
y_M &= \text{r\_nonsep}([y'_{k+1}, \ldots, y'_n], l) \\
y'_{i=1:k} &= z_{i,[0,1]} \\
y'_{i=k+1:n} &= \text{s\_linear}(z_{i,[0,1]}, 0.35)
\end{aligned}
$$

WFG7

Minimize

$$f_{m=1:M}(\mathbf{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$
\begin{aligned}
y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\
y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\
y'_{i=1:k} &= y''_i \\
y'_{i=k+1:n} &= \text{s\_linear}(y''_i, 0.35) \\
y''_{i=1:k} &= \text{b\_param}(z_{i,[0,1]}, \text{r\_sum}([z_{i+1,[0,1]}, \ldots, z_{n,[0,1]}], [1, \ldots, 1]), 0.98/49.98, 0.02, 50) \\
y''_{i=k+1:n} &= z_{i,[0,1]}
\end{aligned}
$$

WFG8

Minimize

$$f_{m=1:M}(\mathbf{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$
\begin{aligned}
y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\
y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\
y'_{i=1:k} &= y''_i \\
y'_{i=k+1:n} &= \text{s\_linear}(y''_i, 0.35) \\
y''_{i=1:k} &= z_{i,[0,1]} \\
y''_{i=k+1:n} &= \text{b\_param}(z_{i,[0,1]}, \text{r\_sum}([z_{1,[0,1]}, \ldots, z_{i-1,[0,1]}], [1, \ldots, 1]), 0.98/49.98, 0.02, 50)
\end{aligned}
$$

WFG9

Minimize

$$f_{m=1:M}(\mathbf{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$
\begin{aligned}
y_{i=1:M-1} &= \text{r\_nonsep}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], k/(M-1)) \\
y_M &= \text{r\_nonsep}([y'_{k+1}, \ldots, y'_n], l) \\
y'_{i=1:k} &= \text{s\_decept}(y''_i, 0.35, 0.001, 0.05) \\
y'_{i=k+1:n} &= \text{s\_multi}(y''_i, 30, 95, 0.35) \\
y''_{i=1:n-1} &= \text{b\_param}(z_{i,[0,1]}, \text{r\_sum}([z_{i+1,[0,1]}, \ldots, z_{n,[0,1]}], [1, \ldots, 1]), 0.98/49.98, 0.02, 50) \\
y''_n &= z_{n,[0,1]}
\end{aligned}
$$

For all problems:

The decision vector is $z = [z_1, \ldots, z_k, z_{k+1}, \ldots, z_n]$ where $0 \le z_i \le z_{i,\max}$.

$$
\begin{aligned}
z_{i=1:n,\max} &= 2i \\
z_{i=1:n,[0,1]} &= \frac{z_i}{z_{i,\max}} \\
x_{i=1:M-1} &= \max(y_M, A_i)(y_i - 0.5) + 0.5 \\
x_M &= y_M \\
S_{m=1:M} &= 2m \\
A_1 &= 1 \\
A_{2:M-1} &= \begin{cases} 0, \text{ for WFG3} \\ 1, \text{ otherwise} \end{cases}
\end{aligned}
$$

In Figures F.1 to F.18, we show the Pareto fronts for all the previous problems for: $n = 24$ and $M = 3$. In each case, two different views of PF_{true} (obtained by enumeration) are shown. Because of the high dimensionality of these test problems, P_{true} is not shown in graphical form.

A.6 MOP Numeric Constrained Test Functions

Table A.5: MOP Numeric Test Functions (with side constraints)

Researcher & Major MOP Characteristics	Definition	Constraints
Belegundu [100]; P_{true} connected, PF_{true} connected	$F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = -2x + y$, $f_2(x,y) = 2x + y$	$0 \leq x \leq 5,\ 0 \leq y \leq 3$, $0 \geq -x + y - 1$, $0 \geq x + y - 7$
Binh (2) [122]; P_{true} connected, PF_{true} convex	$F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = 4x^2 + 4y^2$, $f_2(x,y) = (x-5)^2 + (y-5)^2$	$0 \leq x \leq 5,\ 0 \leq y \leq 3$, $0 \geq (x-5)^2 + y^2 - 25$, $0 \geq -(x-8)^2 - (y+3)^2 + 7.7$
Binh (4) [124];	$F = (f_1(x,y), f_2(x,y), f_3(x,y))$, where $f_1(x,y) = 1.5 - x(1-y)$, $f_2(x,y) = 2.25 - x(1-y^2)$, $f_3(x,y) = 2.625 - x(1-y^3)$	$-10 \leq x, y \leq 10$, $0 \geq -x^2 - (y-0.5)^2 + 9$, $0 \geq (x-1)^2 + (y-0.5)^2 - 6.25$
Jimenez [768]; P_{true} connected and symmetric, PF_{true} convex	Maximize $F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = 5x + 3y$, $f_2(x,y) = 2x + 8y$	$x, y \geq 0$, $0 \geq x + 4y - 100$, $0 \geq 3x + 2y - 150$, $0 \geq 200 - 5x - 3y$, $0 \geq 75 - 2x - 8y$
Kita [833]; P_{true} disconnected, PF_{true} disconnected and concave	Maximize $F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = -x^2 + y$, $f_2(x,y) = \frac{1}{2}x + y + 1$	$x, y \geq 0$, $0 \geq \frac{1}{6}x + y - \frac{13}{2}$, $0 \geq \frac{1}{2}x + y - \frac{15}{2}$, $0 \geq 5x + y - 30$
Obayashi [1150]; P_{true} connected and symmetric, PF_{true} convex	Maximize $F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = x$, $f_2(x,y) = y$	$0 \leq x, y \leq 1$, $x^2 + y^2 \leq 1$

Table A.5: (continued)

Researcher & Major MOP Characteristics	Definition	Constraints
Osyczka [1177]; P_{true} disconnected, PF_{true} convex	$F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = x + y^2$, $f_2(x,y) = x^2 + y$	$2 \leq x \leq 7,\ 5 \leq y \leq 10$, $0 \leq 12 - x - y$, $0 \leq x^2 + 10x - y^2 + 16y - 80$
Osyczka (2) [1177]; P_{true} disconnected, PF_{true} disconnected	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where $f_1(\mathbf{x}) = -(25(x_1-2)^2 + (x_2-2)^2 + (x_3-1)^2 + (x_4-4)^2 + (x_5-1)^2$, $f_2(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$	$0 \leq x_1, x_2, x_6 \leq 10$, $1 \leq x_3, x_5 \leq 5$, $0 \leq x_4 \leq 6$, $0 \leq x_1 + x_2 - 2$, $0 \leq 6 - x_1 - x_2$, $0 \leq 2 - x_2 + x_1$, $0 \leq 2 - x_1 + 3x_2$, $0 \leq 4 - (x_3-3)^2 - x_4$, $0 \leq (x_5-3)^2 + x_6 - 4$
Srinivas[3] [1451]; P_{true} disconnected, PF_{true} connected	$F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = (x-2)^2 + (y-1)^2 + 2$, $f_2(x,y) = 9x - (y-1)^2$	$-20 \leq x, y \leq 20$, $0 \geq x^2 + y^2 - 225$, $0 \geq x - 3y + 10$
Tamaki [1496]; P_{true} connected, a curved surface, PF_{true} a curved surface	Maximize $F = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$, where $f_1(x,y,z) = x$, $f_2(x,y,z) = y$, $f_3(x,y,z) = z$	$0 \leq x, y, z$, $x^2 + y^2 + z^2 \leq 1$
Tanaka [1513]; P_{true} connected, PF_{true} disconnected and convoluted	Minimize $F = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = x$, $f_2(x,y) = y$	$0 < x, y \leq \pi$, $0 \geq -(x^2) - (y^2) + 1 + 0.1 * \cos(16 \arctan \frac{x}{y})$, $\frac{1}{2} \geq (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2$

[3] Deb uses this function with no side constraints as an example in another paper [354].

Table A.5: (continued)

Researcher & Major MOP Characteristics	**Definition**	**Constraints**
Viennet (4) [1592]; P_{true} connected and unsymmetric, PF_{true} a curved surface	$F = (f_1(x,y), f_2(x,y), f_3(x,y))$, where $f_1(x,y) = \frac{(x-2)^2}{2} + \frac{(y+1)^2}{13} + 3,$ $f_2(x,y) = \frac{(x+y-3)^2}{175} + \frac{(2y-x)^2}{17} - 13,$ $f_3(x,y) = \frac{(3x-2y+4)^2}{8} + \frac{(x-y+1)^2}{27}$ $+15$	$-4 \leq x, y \leq 4,$ $y < -4x + 4,$ $x > -1,$ $y > x - 2$
DTLZ 8 [361];	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$, where: $f_1(\mathbf{x}) = \frac{1}{10}\sum_{i=0}^{10} x_i$ $f_2(\mathbf{x}) = \frac{1}{10}\sum_{i=10}^{20} x_i$ $f_3(\mathbf{x}) = \frac{1}{10}\sum_{i=20}^{30} x_i$	$n = 30,$ $0 \leq x_i \leq 1,$ $i = 1, \ldots, 30$ $1 \leq f_3(\mathbf{x}) + 4f_1(\mathbf{x})$ $1 \leq f_3(\mathbf{x}) + 4f_2(\mathbf{x})$ $1 \leq 2f_3(\mathbf{x}) + f_1(\mathbf{x})$ $+f_2(\mathbf{x})$
DTLZ 9 [361];	$F = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$, where: $f_1(\mathbf{x}) = \frac{1}{10}\sum_{i=0}^{10} x_i^{0.1}$ $f_2(\mathbf{x}) = \frac{1}{10}\sum_{i=10}^{20} x_i^{0.1}$ $f_3(\mathbf{x}) = \frac{1}{10}\sum_{i=20}^{30} x_i^{0.1}$	$n = 30,$ $0 \leq x_i \leq 1,$ $i = 1, \ldots, 30$ $1 \leq f_3^2(\mathbf{x}) + f_1^2(\mathbf{x})$ $1 \leq f_3^2(\mathbf{x}) + f_2^2(\mathbf{x})$